Question (1996 STEP I Q13)

I have a Penny Black stamp which I want to sell to my friend Jim, but we cannot agree a price. So I put the stamp under one of two cups, jumble them up, and let Jim guess which one it is under. If he guesses correctly, I add a third cup, jumble them up, and let Jim guess correctly, adding another cup each time. The price he pays for the stamp is £N, where N is the number of cups present when Jim fails to guess correctly. Find $\mathrm{P}(N = k)$. Show that $\mathrm{E}(N) = \mathrm{e}$ and calculate $\mathrm{Var}(N)$.

$$
\begin{aligned}
\mathbb{P}(N = k) &= \mathbb{P}(\text{guesses } k-1 \text{ correctly then 1 wrong}) \\
&= \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{k-1} \frac{k-1}{k} \\
&= \frac{k-1}{k!} \\
\mathbb{E}(N) &= \sum_{k=2}^{\infty} k \cdot \mathbb{P}(N = k) \\
&= \sum_{k=2}^{\infty} \frac{k(k-1)}{k!} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} = e \\
\mathrm{Var}(N) &= \mathbb{E}(N^2) - \mathbb{E}(N)^2 \\
\mathbb{E}(N^2) &= \sum_{k=2}^{\infty} k^2 \mathbb{P}(N = k) \\
&= \sum_{k=2}^{\infty} \frac{k^2(k-1)}{k!} \\
&= \sum_{k=0}^{\infty} \frac{k+2}{k!} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} + 2\sum_{k=0}^{\infty} \frac{1}{k!} = 3e \\
\Rightarrow \qquad \mathrm{Var}(N) &= 3e - e^2
\end{aligned}
$$