Question (1988 STEP II Q11)

A heavy particle lies on a smooth horizontal table, and is attached to one end of a light inextensible string of length L. The other end of the string is attached to a point P on the circumference of the base of a vertical post which is fixed into the table. The base of the post is a circle of radius a with its centre at a point O on the table. Initially, at time $t = 0$, the string is taut and perpendicular to the line OP. The particle is then struck in such a way that the string starts winding round the post and remains taut. At a later time t, a length $a\theta(t)$ $(< L)$ of the string is in contact with the post. Using cartesian axes with origin O, find the position and velocity vectors of the particle at time t in terms of a, L, θ and $\dot{\theta}$, and hence show that the speed of the particle is $(L - a\theta)\dot{\theta}$. If the initial speed of the particle is v, show that the particle hits the post at a time $L^2/(2av)$.

Question (1988 STEP II Q14)

Two particles of mass M and m $(M > m)$ are attached to the ends of a light rod of length $2l$. The rod is fixed at its midpoint to a point O on a horizontal axle so that the rod can swing freely about O in a vertical plane normal to the axle. The axle rotates about a *vertical* axis through O at a constant angular speed ω such that the rod makes a constant angle α $(0 < \alpha < \frac{1}{2}\pi)$ with the vertical. Show that

$$\omega^2 = \left(\frac{M - m}{M + m}\right) \frac{g}{l \cos\alpha}.$$

Show also that the force of reaction of the rod on the axle is inclined at an angle

$$\tan^{-1}\left[\left(\frac{M - m}{M + m}\right)^2 \tan\alpha\right]$$

with the downward vertical.

Question (1989 STEP II Q14)

One end of a light inextrnsible string of length l is fixed to a point on the upper surface of a thin, smooth, horizontal table-top, at a distance $(l - a)$ from one edge of the table-top. A particle of mass m is fixed to the other end of the string, and held a distance a away from this edge of the table-top, so that the string is horizontal and taut. The particle is then released. Find the tension in the string after the string has rotated through an angle θ, and show that the largest magnitude of the force on the edge of the table top is $8mg/\sqrt{3}$.

Question (1989 STEP III Q12)

A smooth horizontal plane rotates with constant angular velocity Ω about a fixed vertical axis through a fixed point O of the plane. The point A is fixed in the plane and $OA = a$. A particle P lies on the plane and is joined to A by a light rod of length $b(> a)$ freely pivoted at A. Initially OAP is a straight line and P is moving with speed $(a + 2\sqrt{ab})\Omega$ perpendicular to OP in the same sense as Ω. At time t, AP makes an angle ϕ with OA produced. Obtain an expression for the component of the acceleration of P perpendicular to AP in terms of $\dfrac{\mathrm{d}^2\phi}{\mathrm{d}t^2}, \phi, a, b$ and Ω. Hence find $\dfrac{\mathrm{d}\phi}{\mathrm{d}t}$, in terms of ϕ, a, b and Ω, and show that P never crosses OA.

Question (1990 STEP III Q13)

A particle P is projected, from the lowest point, along the smooth inside surface of a fixed sphere with centre O. It leaves the surface when OP makes an angle θ with the upward vertical. Find the smallest angle that must be exceeded by θ to ensure that P will strike the surface below the level of O. *You may find it helpful to find the time at which the particle strikes the sphere.*

Question (1991 STEP III Q12)

A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

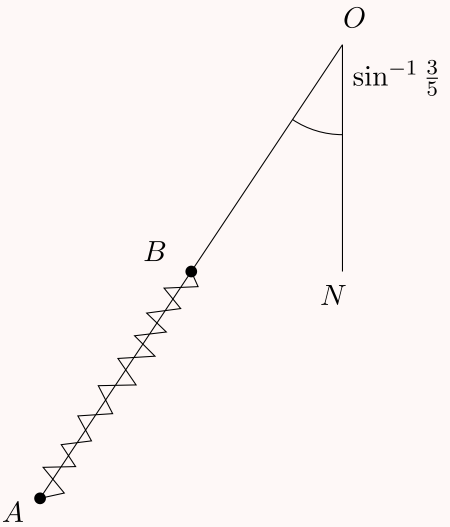
$$x = a\cos\theta \qquad y = b\sin\theta$$

where the x-axis is horizontal and the y-axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section. By first finding $\ddot{x}$ and $\ddot{y}$, or otherwise, show that the acceleration along the inward normal at the point with parameter θ is

$$\frac{ab\dot{\theta}^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

The particle is projected along the surface in the vertical cross-section plane, with speed $2\sqrt{bg}$, from the lowest point. Given that $2a = 3b$, show that it will leave the surface at the point with parameter θ where

$$5\sin^3\theta + 12\sin\theta - 8 = 0.$$



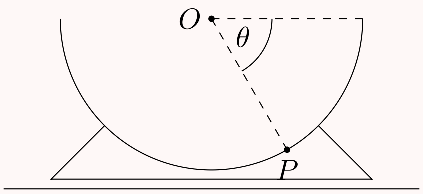
Question (1991 STEP III Q14)

The end O of a smooth light rod OA of length $2a$ is a fixed point. The rod OA makes a fixed angle $\sin^{-1}\frac{3}{5}$ with the downward vertical ON, but is free to rotate about ON. A particle of mass m is attached to the rod at A and a small ring B of mass m is free to slide on the rod but is joined to a spring of natural length a and modulus of elasticity kmg. The vertical plane containing the rod OA rotates about ON with constant angular velocity $\sqrt{5g/2a}$ and B is at rest relative to the rod. Show that the length of OB is

$$\frac{(10k+8)a}{10k-9}.$$

Given that the reaction of the rod on the particle at A makes an angle $\tan^{-1}\frac{13}{21}$ with the horizontal, find the value of k. Find also the magnitude of the reaction between the rod and the ring B.

Question (1992 STEP III Q12)



A smooth hemispherical bowl of mass $2m$ is rigidly mounted on a light carriage which slides freely on a horizontal table as shown in the diagram. The rim of the bowl is horizontal and has centre O. A particle P of mass m is free to slide on the inner surface of the bowl. Initially, P is in contact with the rim of the bowl and the system is at rest. The system is released and when OP makes an angle θ with the horizontal the velocity of the bowl is v? Show that

$$3v = a\dot{\theta}\sin\theta$$

and that

$$v^2 = \frac{2ga\sin^3\theta}{3(3-\sin^2\theta)},$$

where a is the interior radius of the bowl. Find, in terms of m, g and θ, the reaction between the bowl and the particle.

Question (1992 STEP III Q14)

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A horizontal circular disc of radius a and centre O lies on a horizontal table and is fixed to it so that it cannot rotate. A light inextensible string of negligible thickness is wrapped round the disc and attached at its free end to a particle P of mass m. When the string is all in contact with the disc, P is at A. The string is unwound so that the part not in contact with the disc is taut and parallel to OA. P is then at B. The particle is projected along the table from B with speed V perpendicular to and away from OA. In the general position, the string is tangential to the disc at Q and $\angle AOQ = \theta$. Show that, in the general position, the x-coordinate of P with respect to the axes shown in the figure is $a\cos\theta + a\theta\sin\theta$, and find y-coordinate of P. Hence, or otherwise, show that the acceleration of P has components $a\theta\dot{\theta}^2$ and $a\dot{\theta}^2 + a\theta\ddot{\theta}$ along and perpendicular to PQ, respectively.

The friction force between P and the table is $2\lambda m v^2/a$, where v is the speed of P and λ is a constant. Show that

$$\frac{\ddot{\theta}}{\dot{\theta}} = -\left(\frac{1}{\theta} + 2\lambda\theta\right)\dot{\theta}$$

and find $\dot{\theta}$ in terms of θ, λ and a. Find also the tension in the string when $\theta = \pi$.

Question (1996 STEP II Q10)

The plot of 'Rhode Island Red and the Henhouse of Doom' calls for the heroine to cling on to the circumference of a fairground wheel of radius a rotating with constant angular velocity ω about its horizontal axis and then let go. Let ω_0 be the largest value of ω for which it is not possible for her subsequent path to carry her higher than the top of the wheel. Find ω_0 in terms of a and g. If $\omega > \omega_0$ show that the greatest height above the top of the wheel to which she can rise is

$$\frac{a}{2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2.$$

Question (1996 STEP III Q9)

A particle of mass m is at rest on top of a smooth fixed sphere of radius a. Show that, if the particle is given a small displacement, it reaches the horizontal plane through the centre of the sphere at a distance

$$a(5\sqrt{5} + 4\sqrt{2}3)/27$$

from the centre of the sphere. [Air resistance should be neglected.]

Question (1997 STEP I Q10)

The point A is vertically above the point B. A light inextensible string, with a smooth ring P of mass m threaded onto it, has its ends attached at A and B. The plane APB rotates about AB with constant angular velocity ω so that P describes a horizontal circle of radius r and the string is taut. The angle BAP has value θ and the angle ABP has value ϕ. Show that

$$\tan\frac{\phi-\theta}{2} = \frac{g}{r\omega^2}.$$

Find the tension in the string in terms of m, g, r, ω and $\sin\frac{1}{2}(\theta+\phi)$. Deduce from your results that if $r\omega^2$ is small compared with g, then the tension is approximately $\frac{mg}{2}$

Question (2002 STEP III Q10)

A light hollow cylinder of radius a can rotate freely about its axis of symmetry, which is fixed and horizontal. A particle of mass m is fixed to the cylinder, and a second particle, also of mass m, moves on the rough inside surface of the cylinder. Initially, the cylinder is at rest, with the fixed particle on the same horizontal level as its axis and the second particle at rest vertically below this axis. The system is then released. Show that, if θ is the angle through which the cylinder has rotated, then

$$\ddot{\theta} = \frac{g}{2a}\left(\cos\theta - \sin\theta\right),$$

provided that the second particle does not slip. Given that the coefficient of friction is $(3+\sqrt{3})/6$, show that the second particle starts to slip when the cylinder has rotated through 60°.

Question (2004 STEP III Q9)

A circular hoop of radius a is free to rotate about a fixed horizontal axis passing through a point P on its circumference. The plane of the hoop is perpendicular to this axis. The hoop hangs in equilibrium with its centre, O, vertically below P. The point A on the hoop is vertically below O, so that POA is a diameter of the hoop.

A mouse M runs at constant speed u round the rough inner surface of the lower part of the hoop. Show that the mouse can choose its speed so that the hoop remains in equilibrium with diameter POA vertical. Describe what happens to the hoop when the mouse passes the point at which angle $AOM = 2\arctan\mu$, where μ is the coefficient of friction between mouse and hoop.

Question (2005 STEP III Q11)

A horizontal spindle rotates freely in a fixed bearing. Three light rods are each attached by one end to the spindle so that they rotate in a vertical plane. A particle of mass m is fixed to the other end of each of the three rods. The rods have lengths a, b and c, with $a > b > c$ and the angle between any pair of rods is $\frac{2}{3}\pi$. The angle between the rod of length a and the vertical is θ, as shown in the diagram.

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Find an expression for the energy of the system and show that, if the system is in equilibrium, then

$$\tan\theta = -\frac{(b-c)\sqrt{3}}{2a-b-c}\,.$$

Deduce that there are exactly two equilibrium positions and determine which of the two equilibrium positions is stable.

Show that, for the system to make complete revolutions, it must pass through its position of stable equilibrium with an angular velocity of at least

$$\sqrt{\frac{4gR}{a^2+b^2+c^2}}\,,$$

where $2R^2 = (a-b)^2 + (b-c)^2 + (c-a)^2$.

Question (2007 STEP III Q11) **(i)** A wheel consists of a thin light circular rim attached by light spokes of length a to a small hub of mass m. The wheel rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the wheel is vertical throughout the motion. The speed of the wheel is u, where $u^2 < ag$.

Show that, after the wheel reaches the edge of the table and while it is still in contact with the table, the frictional force on the wheel is zero. Show also that the hub will fall a vertical distance $(ag - u^2)/(3g)$ before the rim loses contact with the table.

(ii) Two particles, each of mass $m/2$, are attached to a light circular hoop of radius a, at the ends of a diameter. The hoop rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the hoop is vertical throughout the motion. When the centre of the hoop is vertically above the edge of the table it has speed u, where $u^2 < ag$, and one particle is vertically above the other. Show that, after the hoop reaches the edge of the table and while it is still in contact with the table, the frictional force on the hoop is non-zero and deduce that the hoop will slip before it loses contact with the table.

Question (2008 STEP I Q9)

Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ. Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$. Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

Question (2010 STEP III Q9)

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[tl](6.61,3.14)O [tl](3.31,4.89)P [tl](6.85,4.39)Q [dotsize=6pt 0,dotstyle=*](3.78,4.37)
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The diagram shows two particles, P and Q, connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre O, which is at the edge of the table, and radius a. The angle between OP and the table is θ. The masses of P and Q are m and M, respectively, where $m < M$. Initially, P is held at rest on the table and in contact with the block, Q is vertically above O, and the string is taut. Then P is released. Given that, in the subsequent motion, P remains in contact with the block as θ increases from 0 to $\frac{1}{2}\pi$, find an expression, in terms of m, M, θ and g, for the normal reaction of the block on P and show that

$$\frac{m}{M} \geq \frac{\pi - 1}{3}\,.$$

Question (2011 STEP III Q9)

Particles P and Q have masses $3m$ and $4m$, respectively. They lie on the outer curved surface of a smooth circular cylinder of radius a which is fixed with its axis horizontal. They are connected by a light inextensible string of length $\frac{1}{2}\pi a$, which passes over the surface of the cylinder. The particles and the string all lie in a vertical plane perpendicular to the axis of the cylinder, and the axis intersects this plane at O. Initially, the particles are in equilibrium. Equilibrium is slightly disturbed and Q begins to move downwards. Show that while the two particles are still in contact with the cylinder the angle θ between OQ and the vertical satisfies

$$7a\dot{\theta}^2 + 8g\cos\theta + 6g\sin\theta = 10g\,.$$

(i) Given that Q loses contact with the cylinder first, show that it does so when $\theta = \beta$, where β satisfies

$$15\cos\beta + 6\sin\beta = 10.$$

(ii) Show also that while P and Q are still in contact with the cylinder the tension in the string is $\frac{12}{7}mg(\sin\theta + \cos\theta)$.

Question (2011 STEP III Q11)

A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc. Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b\sin\phi = 2a\sin\tfrac{1}{2}\theta\,.$$

Show further that the magnitude of the couple is

$$\frac{mga^2\sin\theta}{\sqrt{b^2 - 4a^2\sin^2\frac{1}{2}\theta}}\,.$$

The disc is now released from rest. Show that its angular speed, ω, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2\sin^2\tfrac{1}{2}\theta}\,.$$

Question (2012 STEP III Q10)

A small ring of mass m is free to slide without friction on a hoop of radius a. The hoop is fixed in a vertical plane. The ring is connected by a light elastic string of natural length a to the highest point of the hoop. The ring is initially at rest at the lowest point of the hoop and is then slightly displaced. In the subsequent motion the angle of the string to the downward vertical is ϕ. Given that the ring first comes to rest just as the string becomes slack, find an expression for the modulus of elasticity of the string in terms of m and g.

Show that, throughout the motion, the magnitude R of the reaction between the ring and the hoop is given by

$$R = (12\cos^2\phi - 15\cos\phi + 5)mg$$

and that R is non-zero throughout the motion.

Question (2015 STEP II Q9)

An equilateral triangle ABC is made of three light rods each of length a. It is free to rotate in a vertical plane about a horizontal axis through A. Particles of mass $3m$ and $5m$ are attached to B and C respectively. Initially, the system hangs in equilibrium with BC below A.

(i) Show that, initially, the angle θ that BC makes with the horizontal is given by $\sin\theta = \frac{1}{7}$.

(ii) The triangle receives an impulse that imparts a speed v to the particle B. Find the minimum speed v_0 such that the system will perform complete rotations if $v > v_0$.

Question (2016 STEP III Q10)

A smooth plane is inclined at an angle α to the horizontal. A particle P of mass m is attached to a fixed point A above the plane by a light inextensible string of length a. The particle rests in equilibrium on the plane, and the string makes an angle β with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of u. Show that the particle will not immediately leave the plane if $ag\cos(\alpha+\beta) > u^2\tan\beta$.

Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $6\tan\alpha\tan\beta < 1$.

Question (2018 STEP III Q11)

A particle is attached to one end of a light inextensible string of length b. The other end of the string is attached to a fixed point O. Initially the particle hangs vertically below O. The particle then receives a horizontal impulse. The particle moves in a circular arc with the string taut until the acute angle between the string and the upward vertical is α, at which time it becomes slack. Express V, the speed of the particle when the string becomes slack, in terms of b, g and α.

Show that the string becomes taut again a time T later, where

$$gT = 4V \sin\alpha\,,$$

and that just before this time the trajectory of the particle makes an angle β with the horizontal where $\tan\beta = 3\tan\alpha$. When the string becomes taut, the momentum of the particle in the direction of the string is destroyed. Show that the particle comes instantaneously to rest at this time if and only if

$$\sin^2\alpha = \frac{1+\sqrt{3}}{4}\,.$$

Question (1987 STEP I Q11)

A smooth sphere of radius r stands fixed on a horizontal floor. A particle of mass m is displaced gently from equilibrium on top of the sphere. Find the angle its velocity makes with the horizontal when it loses contact with the sphere during the subsequent motion. By energy considerations, or otherwise, find the vertical component of the momentum of the particle as it strikes the floor.

Question (1987 STEP III Q12)

A firework consists of a uniform rod of mass M and length $2a$, pivoted smoothly at one end so that it can rotate in a fixed horizontal plane, and a rocket attached to the other end. The rocket is a uniform rod of mass $m(t)$ and length $2l(t)$, with $m(t) = 2\alpha l(t)$ and α constant. It is attached to the rod by its front end and it lies at right angles to the rod in the rod's plane of rotation. The rocket burns fuel in such a way that $\mathrm{d}m/\mathrm{d}t = -\alpha\beta$, with β constant. The burnt fuel is ejected from the back of the rocket, with speed u and directly backwards relative to the rocket. Show that, until the fuel is exhausted, the firework's angular velocity ω at time t satisfies

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3\alpha\beta a u}{2[Ma^2 + 2\alpha l(3a^2 + l^2)]}.$$

Question (2019 STEP III Q9)

In this question, $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors and $\mathbf{j}$ is vertically upwards. A smooth hemisphere of mass M and radius a rests on a smooth horizontal table with its plane face in contact with the table. The point A is at the top of the hemisphere and the point O is at the centre of its plane face. Initially, a particle P of mass m rests at A. It is then given a small displacement in the positive $\mathbf{i}$ direction. At a later time t, when the particle is still in contact with the hemisphere, the hemisphere has been displaced by $-s\mathbf{i}$ and $\angle AOP = \theta$.

(i) Let $\mathbf{r}$ be the position vector of the particle at time t with respect to the initial position of O. Write down an expression for $\mathbf{r}$ in terms of a, θ and s and show that

$$\dot{\mathbf{r}} = (a\dot{\theta}\cos\theta - \dot{s})\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}.$$

Show also that

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta,$$

where $k = \frac{M}{m+M}$, and deduce that

$$\dot{\mathbf{r}} = a\dot{\theta}(k\cos\theta\mathbf{i} - \sin\theta\mathbf{j}).$$

(ii) Show that

$$a\dot{\theta}^2\left(k\cos^2\theta + \sin^2\theta\right) = 2g(1-\cos\theta).$$

(iii) At time T, when $\theta = \alpha$, the particle leaves the hemisphere. By considering the component of $\ddot{\mathbf{r}}$ parallel to the vector $\sin\theta\mathbf{i} + k\cos\theta\mathbf{j}$, or otherwise, show that at time T

$$a\dot{\theta}^2 = g\cos\alpha.$$

Find a cubic equation for $\cos\alpha$ and deduce that $\cos\alpha > \frac{2}{3}$.

Question (2025 STEP II Q9)

Points A and B are at the same height and a distance $\sqrt{2}r$ apart. Two small, spherical particles of equal mass, P and Q, are suspended from A and B, respectively, by light inextensible strings of length r. Each particle individually may move freely around and inside a circle centred at the point of suspension. The particles are projected simultaneously from points which are a distance r vertically below their points of suspension, directly towards each other and each with speed u. When the particles collide, the coefficient of restitution in the collision is e.

(i) Show that, immediately after the collision, the horizontal component of each particle's velocity has magnitude $\frac{1}{2}ev\sqrt{2}$, where $v^2 = u^2 - gr(2-\sqrt{2})$ and write down the vertical component in terms of v.

(ii) Show that the strings will become taut again at a time t after the collision, where t is a non-zero root of the equation

$$(r - evt)^2 + \left(-r + vt - \frac{1}{2}\sqrt{2}gt^2\right)^2 = 2r^2.$$

(iii) Show that, in terms of the dimensionless variables

$$z = \frac{vt}{r} \quad \text{and} \quad c = \frac{\sqrt{2}v^2}{rg}$$

this equation becomes

$$\left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(1+e) = 0.$$

(iv) Show that, if this equation has three equal non-zero roots, $e = \frac{1}{3}$ and $v^2 = \frac{9}{2}\sqrt{2}rg$. Explain briefly why, in this case, no energy is lost when the string becomes taut.

(v) In the case described in (iv), the particles have speed U when they again reach the points of their motion vertically below their points of suspension. Find U^2 in terms of r and g.