Question (1988 STEP II Q2)

The numbers x, y and z are non-zero, and satisfy

$$2a - 3y = \frac{(z-x)^2}{y} \quad \text{and} \quad 2a - 3z = \frac{(x-y)^2}{z},$$

for some number a. If $y \neq z$, prove that

$$x + y + z = a,$$

and that

$$2a - 3x = \frac{(y-z)^2}{x}.$$

Determine whether this last equation holds *only* if $y \neq z$.

Question (1991 STEP II Q1)

Let $\mathrm{h}(x) = ax^2 + bx + c$, where a, b and c are constants, and $a \neq 0$. Give a condition which a, b and c must satisfy in order that $\mathrm{h}(x)$ can be written in the form

$$a(x+k)^2, \qquad (*)$$

where k is a constant. If $\mathrm{f}(x) = 3x^2 + 4x$ and $\mathrm{g}(x) = x^2 - 2$, find the two constant values of λ such that $\mathrm{f}(x) + \lambda\mathrm{g}(x)$ can be written in the form $(*)$. Hence, or otherwise, find constants A, B, C, D, m and n such that

$$\mathrm{f}(x) = A(x+m)^2 + B(x+n)^2$$
$$\mathrm{g}(x) = C(x+m)^2 + D(x+n)^2.$$

If $\mathrm{f}(x) = 3x^2 + 4x$ and $\mathrm{g}(x) = x^2 + \alpha$ and it is given by that there is only one value of λ for which $\mathrm{f}(x) + \lambda\mathrm{g}(x)$ can be written in the form $(*)$, find α.

Question (1991 STEP II Q3)

It is given that x, y and z are distinct and non-zero, and that they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Show that $x^2y^2z^2 = 1$ and that the value of $x + \dfrac{1}{y}$ is either $+1$ or -1.

Question (1996 STEP III Q8)

A transformation T of the real numbers is defined by

$$y = T(x) = \frac{ax - b}{cx - d},$$

where $a, b, c,$ d are real numbers such that $ad \neq bc$. Find all numbers x such that $T(x) = x$. Show that the inverse operation, $x = T^{-1}(y)$ expressing x in terms of y is of the same form as T and find corresponding numbers a', b', c',d'.

Let S_r denote the set of all real numbers excluding r. Show that, if $c \neq 0$, there is a value of r such that T is defined for all $x \in S_r$ and find the image $T(S_r)$. What is the corresponding result if $c = 0$?

If T_1, given by numbers a_1, b_1, c_1, d_1, and T_2, given by numbers a_2, b_2, c_2, d_2 are two such transformations, show that their composition T_3, defined by $T_3(x) = T_2(T_1(x))$, is of the same form.

Find necessary and sufficient conditions on the numbers a, b, c, d for T^2, the composition of T with itself, to be the identity. Hence, or otherwise, find transformations T_1, T_2 and their composition T_3 such that T_1^2 and T_2^2 are each the identity but T_3^2 is not.

Question (2009 STEP II Q1)

Two curves have equations $x^4 + y^4 = u$ and $xy = v$, where u and v are positive constants. State the equations of the lines of symmetry of each curve. The curves intersect at the distinct points A, B, C and D (taken anticlockwise from A). The coordinates of A are (α, β), where $\alpha > \beta > 0$. Write down, in terms of α and β, the coordinates of B, C and D. Show that the quadrilateral $ABCD$ is a rectangle and find its area in terms of u and v only. Verify that, for the case $u = 81$ and $v = 4$, the area is 14.

Question (2010 STEP I Q1)

Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d. Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

Question (2012 STEP III Q3)

It is given that the two curves

$$y = 4 - x^2 \text{ and } mx = k - y^2\,,$$

where $m > 0$, touch exactly once.

(i) In each of the following four cases, sketch the two curves on a single diagram, noting the coordinates of any intersections with the axes:

a) $k < 0\,;$

b) $0 < k < 16$, $k/m < 2\,;$

c) $k > 16$, $k/m > 2\,;$

d) $k > 16$, $k/m < 2\,.$

(ii) Now set $m = 12$. Show that the x-coordinate of any point at which the two curves meet satisfies

$$x^4 - 8x^2 + 12x + 16 - k = 0\,.$$

Let a be the value of x at the point where the curves touch. Show that a satisfies

$$a^3 - 4a + 3 = 0$$

and hence find the three possible values of a. Derive also the equation

$$k = -4a^2 + 9a + 16\,.$$

Which of the four sketches in part (i) arise?

Question (2016 STEP III Q8) **(i)** The function f satisfies, for all x, the equation

$$f(x) + (1-x)f(-x) = x^2\,.$$

Show that $f(-x) + (1+x)f(x) = x^2\,.$ Hence find $f(x)$ in terms of x. You should verify that your function satisfies the original equation.

(ii) The function K is defined, for $x \neq 1$, by

$$\mathrm{K}(x) = \frac{x+1}{x-1}\,.$$

Show that, for $x \neq 1$, $\mathrm{K}(K(x)) = x\,.$

The function g satisfies the equation

$$g(x) + x\,g\Big(\frac{x+1}{x-1}\Big) = x \qquad (x \neq 1)\,.$$

Show that, for $x \neq 1$, $g(x) = \dfrac{2x}{x^2+1}\,.$

(iii) Find $h(x)$, for $x \neq 0$, $x \neq 1$, given that

$$h(x) + h\Big(\frac{1}{1-x}\Big) = 1 - x - \frac{1}{1-x} \qquad (x \neq 0, \quad x \neq 1)\,.$$