Question (1995 STEP III Q14)

A candidate finishes examination questions in time T, where T has probability density function

$$\mathrm{f}(t) = t\mathrm{e}^{-t} \qquad t \geqslant 0,$$

the probabilities for the various questions being independent. Find the moment generating function of T and hence find the moment generating function for the total time U taken to finish two such questions. Show that the probability density function for U is

$$\mathrm{g}(u) = \frac{1}{6}u^3\mathrm{e}^{-u} \qquad u \geqslant 0.$$

Find the probability density function for the total time taken to answer n such questions.

Question (1997 STEP III Q13)

Let X and Y be independent standard normal random variables: the probability density function, f, of each is therefore given by

$$f(x) = (2\pi)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}x^2}.$$

(i) Find the moment generating function $\mathrm{E}(\mathrm{e}^{\theta X})$ of X.

(ii) Find the moment generating function of $aX+bY$ and hence obtain the condition on a and b which ensures that $aX+bY$ has the same distribution as X and Y.

(iii) Let $Z = \mathrm{e}^{\mu+\sigma X}$. Show that

$$\mathrm{E}(Z^\theta) = \mathrm{e}^{\mu\theta+\frac{1}{2}\sigma^2\theta^2},$$

and hence find the expectation and variance of Z.

Question (1999 STEP III Q12)

In the game of endless cricket the scores X and Y of the two sides are such that

$$\mathbb{P}(X = j,\ Y = k) = \mathrm{e}^{-1}\frac{(j+k)\lambda^{j+k}}{j!k!},$$

for some positive constant λ, where $j, k = 0,\ 1,\ 2,\ \ldots$.

(i) Find $\mathbb{P}(X + Y = n)$ for each $n > 0$.

(ii) Show that $2\lambda\mathrm{e}^{2\lambda-1} = 1$.

(iii) Show that $2x\mathrm{e}^{2x-1}$ is an increasing function of x for $x > 0$ and deduce that the equation in (ii) has at most one solution and hence determine λ.

(iv) Calculate the expectation $\mathbb{E}(2^{X+Y})$.

Question (2008 STEP III Q12)

Let X be a random variable with a Laplace distribution, so that its probability density function is given by

$$f(x) = \frac{1}{2}\mathrm{e}^{-|x|}\ , \qquad -\infty < x < \infty\ . \qquad (*)$$

Sketch $f(x)$. Show that its moment generating function $\mathrm{M}_X(\theta)$ is given by $\mathrm{M}_X(\theta) = (1-\theta^2)^{-1}$ and hence find the variance of X. A frog is jumping up and down, attempting to land on the same spot each time. In fact, in each of n successive jumps he always lands on a fixed straight line but when he lands from the ith jump $(i = 1\,, 2\,, \ldots\,, n)$ his displacement from the point from which he jumped is X_i cm, where X_i has the distribution $(*)$. His displacement from his starting point after n jumps is Y cm (so that $Y = \sum\limits_{i=1}^{n} X_i$). Each jump is independent of the others. Obtain the moment generating function for $Y/\sqrt{2n}$ and, by considering its logarithm, show that this moment generating function tends to $\exp(\frac{1}{2}\theta^2)$ as $n \to \infty$. Given that $\exp(\frac{1}{2}\theta^2)$ is the moment generating function of the standard Normal random variable, estimate the least number of jumps such that there is a 5% chance that the frog lands 25 cm or more from his starting point.