Question (1988 STEP II Q8)

In a crude model of population dynamics of a community of aardvarks and buffaloes, it is assumed that, if the numbers of aardvarks and buffaloes in any year are A and B respectively, then the numbers in the following year at $\frac{1}{4}A + \frac{3}{4}B$ and $\frac{3}{2}B - \frac{1}{2}A$ respectively. It does not matter if the model predicts fractions of animals, but a non-positive number of buffaloes means that the species has become extinct, and the model ceases to apply. Using matrices or otherwise, show that the ratio of the number of aardvarks to the number of buffaloes can remain the same each year, provided it takes one of two possible values. Let these two possible values be x and y, and let the numbers of aardvarks and buffaloes in a given year be a and b respectively. By writing the vector (a, b) as a linear combination of the vectors $(x, 1)$ and $(y, 1)$, or otherwise, show how the numbers of aardvarks and buffaloes in subsequent years may be found. On a sketch of the a-b plane, mark the regions which correspond to the following situations

(i) an equilibrium population is reached as time $t \to \infty$;

(ii) buffaloes become extinct after a finite time;

(iii) buffaloes approach extinction as $t \to \infty$.

If the population in a given year is $\mathbf{v} = \begin{pmatrix} A \\ B \end{pmatrix}$ then the population the next year is $\mathbf{Mv}$ where $\mathbf{M} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

The ratio is the same if $\mathbf{Mv} = \lambda \mathbf{v}$ ie if $\mathbf{v}$ is an eigenvector of $\mathbf{M}$.

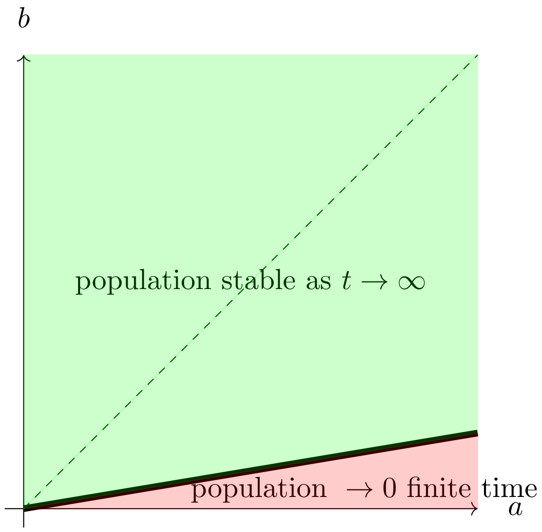
The eigenvalues will be 1 and $\frac{3}{8}$ (by inspection) so we should be able to solve for the eigenvectors:

$\lambda = 1$ we have $\frac{1}{4}A + \frac{3}{4}B = A \Rightarrow A = B$ a ratio of 1. $\lambda = \frac{3}{8}$ we have $\frac{1}{4}A + \frac{3}{4}B = \frac{3}{8}A \Rightarrow \frac{3}{4}B = \frac{1}{8}A \Rightarrow A = 6B$ a ratio of 6.

If we write $\begin{pmatrix} a \\ b \end{pmatrix}$ as $x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_6 \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ we find that after n years, we have:

$x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\frac{3}{8}\right)^n x_6 \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ for the populations.

Therefore if x_1 is < 0 then in finite time we will end up with one population being 0. If $x_1 > 0$ are positive we tend to a finite population and if $x_1 = 0$ then over time the population will tend to 0 at infinity. In our diagram these areas correspond to (red) - die out in finite time, (green) population stable and the thick black line where the population goes extinct as $t \to \infty$



Question (1990 STEP II Q10)

Two square matrices $\mathbf{A}$ and $\mathbf{B}$ satisfies $\mathbf{AB} = \mathbf{0}$. Show that either $\det \mathbf{A} = 0$ or $\det \mathbf{B} = 0$ or $\det \mathbf{A} = \det \mathbf{B} = 0$. If $\det \mathbf{B} \neq 0$, what must $\mathbf{A}$ be? Give an example to show that the condition $\det \mathbf{A} = \det \mathbf{B} = 0$ is not sufficient for the equation $\mathbf{AB} = \mathbf{0}$ to hold. Find real numbers p, q and r such that

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = (\mathbf{M} + p\mathbf{I})(\mathbf{M} + q\mathbf{I})(\mathbf{M} + r\mathbf{I}),$$

where $\mathbf{M}$ is any square matrix and $\mathbf{I}$ is the appropriate identity matrix. Hence, or otherwise, find all matrices $\mathbf{M}$ of the form $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ which satisfy the equation

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = \mathbf{0}.$$

Since $0 = \det \mathbf{0} = \det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ at least one of $\det \mathbf{A}$ or $\det \mathbf{B}$ is zero.

If $\det \mathbf{B} \neq 0$ then $\mathbf{B}$ is invertible, and multiplying on the right by $\mathbf{B}^{-1}$ gives us $\mathbf{A} = \mathbf{0}$.

If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, then $\det \mathbf{A} = \det \mathbf{B} = 0$, but $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0}$

Since $\mathbf{M}$ commutes with itself and the identity matrix, this is equivalent to factorising the polynomial over the reals. Therefore

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = (\mathbf{M} - 2\mathbf{I})(\mathbf{M} + \mathbf{I})(\mathbf{M} + 3\mathbf{I}),$$

Since we now know at least one of $\det(\mathbf{M} - 2\mathbf{I})$, $\det(\mathbf{M} + \mathbf{I})$, $\det(\mathbf{M} + 3\mathbf{I})$, we should look at cases:

Since at least one of those must be non-zero, we must have the following cases: $(a, b) = (2, -1), (-1, 2), (-1, -3), (-3, -1), (2, -3), (-3, 2)$

In each of those cases, we will have:

$\begin{pmatrix} 0 & c \\ 0 & b + k \end{pmatrix} \begin{pmatrix} a + l & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so all of those solutions are valid. So c can be anything as long as (a, b) are in that set of solutions

Question (1992 STEP III Q2)

The matrices $\mathbf{I}$ and $\mathbf{J}$ are

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

respectively and $\mathbf{A} = \mathbf{I} + a\mathbf{J}$, where a is a non-zero real constant. Prove that

$$\mathbf{A}^2 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^2 - 1]\mathbf{J} \quad \text{and} \quad \mathbf{A}^3 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^3 - 1]\mathbf{J}$$

and obtain a similar form for $\mathbf{A}^4$.

If $\mathbf{A}^k = \mathbf{I} + p_k\mathbf{J}$, suggest a suitable form for p_k and prove that it is correct by induction, or otherwise.

If $\mathbf{J} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, them $\mathbf{J}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2\mathbf{J}$. Therefore $\mathbf{J}^n = 2\mathbf{J}^{n-1} = 2^{n-1}\mathbf{J}$

Let $\mathbf{A} = \mathbf{I} + a\mathbf{J}$ then

$$\begin{aligned} \mathbf{A}^2 &= (\mathbf{I} + a\mathbf{J})^2 \\ &= \mathbf{I} + 2a\mathbf{J} + a^2\mathbf{J}^2 \\ &= \mathbf{I} + 2a\mathbf{J} + 2a^2\mathbf{J} \\ &= \mathbf{I} + (2a + 2a^2)\mathbf{J} \\ &= \mathbf{I} + \frac{1}{2}(1 + 4a + 4a^2 - 1)\mathbf{J} \\ &= \mathbf{I} + \frac{1}{2}((1+2a)^2 - 1)\mathbf{J} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^3 &= (\mathbf{I} + a\mathbf{J})^3 \\ &= \mathbf{I} + 3a\mathbf{J} + a^2\mathbf{J} + a^3\mathbf{J}^3 \\ &= \mathbf{I} + 3a\mathbf{J} + 6a^2\mathbf{J} + 4a^3\mathbf{J} \\ &= \mathbf{I} + (3a + 6a^3 + 4a^3)\mathbf{J} \\ &= \mathbf{I} + \frac{1}{2}(1 + 3\cdot 2a + 3\dot{4}a^2 + 8a^3 - 1)\mathbf{J} \\ &= \mathbf{I} + \frac{1}{2}((1+2a)^3 - 1)\mathbf{J} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^4 &= (\mathbf{I} + a\mathbf{J})^4 \\ &= \mathbf{I} + 4a\mathbf{J} + 6a^2\mathbf{J}^2 + 4a^3\mathbf{J}^3 + a^4\mathbf{J}^4 \\ &= \mathbf{I} + 4a\mathbf{J} + 12a^2\mathbf{J} + 16a^3\mathbf{J} + 8a^4\mathbf{J} \\ &= \mathbf{I} + (4a + 12a^3 + 16a^3 + 8a^4)\mathbf{J} \\ &= \mathbf{I} + \frac{1}{2}(1 + 4\cdot 2a + 6\cdot 4a^2 + 4\cdot 8a^3 + 16a^4 - 1)\mathbf{J} \\ &= \mathbf{I} + \frac{1}{2}((1+2a)^4 - 1)\mathbf{J} \end{aligned}$$

Claim: $\mathbf{A}^k = \mathbf{I} + \frac{1}{2}((1+2a)^k - 1)\mathbf{J}$

Proof: Firstly, note that $\mathbf{I}$ commutes with everything, so we can just apply the binomial theorem as if we were using real numbers:

$$\begin{aligned}
\mathbf{A}^k &= (\mathbf{I} + a\mathbf{J})^k \\
&= \sum_{i=0}^{k} \binom{k}{i} a^i \mathbf{J}^i \\
&= \mathbf{I} + \sum_{i=1}^{k} \binom{k}{i} a^i 2^{i-1} \mathbf{J} \\
&= \mathbf{I} + \frac{1}{2}\left(\sum_{i=1}^{k} \binom{k}{i} a^i 2^i\right) \mathbf{J} \\
&= \mathbf{I} + \frac{1}{2}\left(\sum_{i=0}^{k} \binom{k}{i} a^i 2^i - 1\right) \mathbf{J} \\
&= \mathbf{I} + \frac{1}{2}\left((1+2a)^k - 1\right) \mathbf{J}
\end{aligned}$$

as required

Question (1993 STEP II Q6)

In this question, $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{X}$ *are non-zero* 2×2 *real matrices.* Are the following assertions true or false? You must provide a proof or a counterexample in each case.

(i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.

(ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.

(iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

(i) This is false, for example let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{AB} &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathbf{BA} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

(ii) This is also false, using the same matrices from part (i), we find:

$$\begin{aligned}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B}) &= \mathbf{A}^2 - \mathbf{BA} + \mathbf{AB} - \mathbf{B}^2 \\ &= \mathbf{A}^2 - \mathbf{B}^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &\neq \mathbf{A}^2 - \mathbf{B}^2\end{aligned}$$

(iii) This is true. Claim: The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.

Proof: ($\Rightarrow$) Suppose $\det \mathbf{A} \neq 0$ then $\mathbf{A}$ has an inverse, and so we must have $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{0} \Rightarrow \mathbf{X} = \mathbf{0}$.

($\Leftarrow$) Suppose $\det \mathbf{A} = 0$ then $ad - bc = 0$, so consider the matrix $\mathbf{X} = \begin{pmatrix} d & d \\ -c & -c \end{pmatrix}$ (or if this is zero, $\mathbf{X} = \begin{pmatrix} a & a \\ -b & -b \end{pmatrix}$)

(iv) This is false. Consider $\mathbf{A} = \mathbf{B} = \mathbf{0}$, then $\mathbf{X} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ has the property that $\mathbf{X}^2 = \mathbf{0}$ for all x, so at least more than 2 values

Question (1993 STEP II Q10)

Verify that if

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} -1 & 8 \\ 8 & 11 \end{pmatrix}$$

then $\mathbf{PAP}$ is a diagonal matrix. Put $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. By writing

$$\mathbf{x} = \mathbf{Px}_1 + \mathbf{a}$$

for a suitable vector $\mathbf{a}$, show that the equation

$$\mathbf{x}^{\mathrm{T}}\mathbf{Ax} + \mathbf{b}^{\mathrm{T}}\mathbf{x} - 11 = 0,$$

where $\mathbf{b} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, becomes

$$3x_1^2 - y_1^2 = c$$

for some constant c (which you should find).

$$\begin{aligned}\mathbf{PAP} &= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 8 \\ 8 & 11 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 15 & -10 \\ 30 & 5 \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} 75 & 0 \\ 0 & -25 \end{pmatrix}$$

Which is diagonal as required.

Letting $\mathbf{x} = \mathbf{P}\mathbf{x}_1 + \mathbf{a}$

$$\begin{aligned}
& \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} - 11 = 0 \\
\Leftrightarrow \quad & (\mathbf{P}\mathbf{x}_1 + \mathbf{a})^T\mathbf{A}(\mathbf{P}\mathbf{x}_1 + \mathbf{a}) + \mathbf{b}^T(\mathbf{P}\mathbf{x}_1 + \mathbf{a}) - 11 = 0 \\
\Leftrightarrow \quad & \mathbf{x}_1^T\mathbf{PAP}\mathbf{x}_1 + \mathbf{x}_1^T\mathbf{PAa} + \mathbf{a}^T\mathbf{AP}\mathbf{x}_1 + \mathbf{a}^T\mathbf{Aa} + \mathbf{b}^T(\mathbf{P}\mathbf{x}_1 + \mathbf{a}) - 11 = 0 \\
\Leftrightarrow \quad & \mathbf{x}_1^T\mathbf{PAP}\mathbf{x}_1 + (2\mathbf{a}^T\mathbf{A} + \mathbf{b}^T)\mathbf{P}\mathbf{x}_1 + \mathbf{a}^T\mathbf{Aa} + \mathbf{b}^T\mathbf{a} - 11 = 0
\end{aligned}$$

It would be nice if we picked $\mathbf{a}$ such that $2\mathbf{a}^T\mathbf{A} + \mathbf{b}^T = 0$, if $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then this equation becomes:

$$\begin{aligned}
& 2\begin{pmatrix} -a_1 + 8a_2 & 8a_1 + 11a_2 \end{pmatrix} + \begin{pmatrix} 18 & 6 \end{pmatrix} = 0 \\
\Rightarrow \quad & a_1 = 1, a_2 = -1
\end{aligned}$$

So our equation is now

$$\begin{aligned}
& \mathbf{x}_1^T\mathbf{PAP}\mathbf{x}_1 + (2\mathbf{a}^T\mathbf{A} + \mathbf{b}^T)\mathbf{P}\mathbf{x}_1 + \mathbf{a}^T\mathbf{Aa} + \mathbf{b}^T\mathbf{a} - 11 = 0 \\
\Leftrightarrow \quad & \mathbf{x}_1^T\mathbf{PAP}\mathbf{x}_1 - 6 + 12 - 11 = 0 \\
\Leftrightarrow \quad & 25(3x_1^2 - y_1^2) = 5 \\
\Leftrightarrow \quad & 3x_1^2 - y_1^2 = \frac{1}{5}
\end{aligned}$$

Question (2013 STEP I Q3)

For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points $P_1, P_2, \ldots$ are defined by $P_1 = X * Y$ and, for $n \geq 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

(i) Suppose $X * Y = Y * X$, then

$$\begin{aligned} X * Y &= \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \\ Y * X &= \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \\ \Rightarrow \qquad 0 &= (2\lambda - 1)(\mathbf{x} - \mathbf{y}) \end{aligned}$$

Therefore, either $\mathbf{x} = \mathbf{y}$ or $\lambda = \frac{1}{2}$. Since we assumed X, Y were distinct, $\mathbf{x} \neq \mathbf{y}$ and so $X * Y$ and $Y * X$ are distinct unless $\lambda = \frac{1}{2}$

(ii) Suppose $(X * Y) * Z = X * (Y * Z)$

$$\begin{aligned} (X * Y) * Z &= (\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) * \mathbf{z} \\ &= (\lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z} \\ X * (Y * Z) &= \mathbf{x} * (\lambda\mathbf{y} + (1-\lambda)\mathbf{z}) \\ &= (\lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z} \\ \Rightarrow \qquad 0 &= (\lambda^2 - \lambda)\mathbf{x} + ((1-\lambda) - (1-\lambda)^2)\mathbf{z} \\ &= (1-\lambda)(-\lambda\mathbf{x} + \lambda\mathbf{z}) \\ &= \lambda(1-\lambda)(\mathbf{z} - \mathbf{x}) \end{aligned}$$

Therefore they are distinct unless $\lambda = 1, 0$ or $\mathbf{x} = \mathbf{z}$.

(iii) Claim: $(X * Y) * Z = (X * Z) * (Y * Z)$

Proof:

$$\begin{aligned} (X * Y) * Z &= (\lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z} \\ (X * Z) * (Y * Z) &= (\lambda\mathbf{x} + (1-\lambda)\mathbf{z}) * (\lambda\mathbf{y} + (1-\lambda)\mathbf{z}) \\ &= \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{z}) + (1-\lambda)(\lambda\mathbf{y} + (1-\lambda)\mathbf{z}) \\ &= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z} \end{aligned}$$

Claim: $X * (Y * Z) = (X * Y) * (X * Z)$

Proof:

$$\begin{aligned} X * (Y * Z) &= \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z} \\ (X * Y) * (X * Z) &= (\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) * (\lambda\mathbf{x} + (1-\lambda)\mathbf{z}) \\ &= \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)(\lambda\mathbf{x} + (1-\lambda)\mathbf{z}) \\ &= \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z} \end{aligned}$$

(iv) $P_1 = X * Y$ divides the line segment into the ratio $\lambda : (1-\lambda)$. P_n divides the line segment $P_{n-1}Y$ into the ratio $\lambda : (1-\lambda)$, therefore it divides the line segment XY in the ratio $\lambda^n : 1 - \lambda^n$

Alternatively,

$$\begin{aligned} P_1 &= \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \\ P_2 &= (\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) * \mathbf{y} \end{aligned}$$

$$= \lambda^2\mathbf{x} + (1-\lambda^2)\mathbf{y}$$

Suppose $P_k = \lambda^k\mathbf{x} + (1-\lambda^k)\mathbf{y}$ then

$$\begin{aligned} P_{k+1} &= (\lambda^k\mathbf{x} + (1-\lambda^k)\mathbf{y}) * \mathbf{y} \\ &= \lambda^{k+1}\mathbf{x} + \lambda(1-\lambda^k)\mathbf{y} + (1-\lambda)\mathbf{y} \\ &= \lambda^{k+1}\mathbf{x} + (1-\lambda^{k+1})\mathbf{y} \end{aligned}$$

Question (1987 STEP II Q9)

For any square matrix $\mathbf{A}$ such that $\mathbf{I}-\mathbf{A}$ is non-singular (where $\mathbf{I}$ is the unit matrix), the matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}.$$

Prove that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$ if and only if $\mathbf{A}+\mathbf{A}^{\mathrm{T}} = \mathbf{O}$ (where $\mathbf{O}$ is the zero matrix), explaining clearly each step of your proof.

[*You may quote standard results about matrices without proof.*]

We use the following properties: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$

$$\begin{aligned} && \mathbf{I} &= \mathbf{B}^{\mathrm{T}}\mathbf{B} \\ \Leftrightarrow && \mathbf{B}^{T^{-1}} &= \mathbf{B} \\ \Leftrightarrow && (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1} &= ((\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1})^{T^{-1}} \\ &&&= ((\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1})^{-1^T} \\ &&&= ((\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1})^T \\ &&&= (\mathbf{I}+\mathbf{A})^{-1^T}(\mathbf{I}-\mathbf{A})^T \\ &&&= \left(\mathbf{I}+\mathbf{A}^T\right)^{-1}(\mathbf{I}-\mathbf{A}^T) \\ \Leftrightarrow && (\mathbf{I}+\mathbf{A}^T)(\mathbf{I}+\mathbf{A}) &= (\mathbf{I}-\mathbf{A}^T)(\mathbf{I}-\mathbf{A}) \\ \Leftrightarrow && \mathbf{I}+\mathbf{A}+\mathbf{A}^T+\mathbf{A}^T\mathbf{A} &= \mathbf{I}-\mathbf{A}-\mathbf{A}^T+\mathbf{A}^T\mathbf{A} \\ \Leftrightarrow && 2(\mathbf{A}^T+\mathbf{A}) &= \mathbf{O} \\ \Leftrightarrow && \mathbf{A}+\mathbf{A}^T &= \mathbf{O} \end{aligned}$$

Question (2019 STEP II Q8)

The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if M is a 2×2 matrix, then $f(M) \in \mathbb{R}$. The function f has the property that $f(MN) = f(M)f(N)$ for any 2×2 matrices M and N.

(i) You are given that there is a matrix M such that $f(M) \neq 0$. Let I be the 2×2 identity matrix. By considering $f(MI)$, show that $f(I) = 1$.

(ii) Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $f(J) \neq 1$. By considering J^2, evaluate $f(J)$. Using J, show that, for any real numbers a, b, c and d,

$$.f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right)$$

(iii) Let $K = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where $k \in \mathbb{R}$. Use K to show that, if the second row of the matrix A is a multiple of the first row, then $f(A) = 0$.

(iv) Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices P^2, P^{-1}, and $K^{-1}PK$ for suitable values of k, evaluate $f(P)$.

(i) Consider $f(M) = f(MI) = f(M)f(I)$. Since $f(M) \neq 0$ we can divide by $f(M)$ to obtain $f(I) = 1$

(ii) Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $J^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Therefore $1 = f(I) = f(J^2) = f(J)f(J) \Rightarrow f(J) = \pm 1 \Rightarrow f(J) = -1$ since $f(J) \neq 1$.

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} J &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} b & a \\ d & c \end{pmatrix} \\
J\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} c & d \\ a & b \end{pmatrix} \\
J\begin{pmatrix} a & b \\ c & d \end{pmatrix} J &= \begin{pmatrix} d & c \\ b & a \end{pmatrix}
\end{aligned}$$

Therefore $f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(J\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f(J)f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

and

$f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right) = f\left(J\begin{pmatrix} a & b \\ c & d \end{pmatrix}J\right) = f(J)f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)f(J) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ as required.

(iii) First consider O the matrix of 0, then

$$\begin{aligned} & JO = O \\ \Rightarrow \quad & f(JO) = f(O) \\ \Rightarrow \quad & f(J)f(O) = f(O) \\ \Rightarrow \quad & -f(O) = f(O) \\ \Rightarrow \quad & f(O) = 0 \end{aligned}$$

Now consider $K_k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$. Suppose $A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$ then

$$\begin{aligned} K_{\frac{1}{k}}A &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}\begin{pmatrix} a & b \\ ka & kb \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ a & b \end{pmatrix} \end{aligned}$$

And so $f(K_{\frac{1}{k}}A) = f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$, therefore either $f(K_{\frac{1}{k}}) = 0$ or $f(A) = 0$, but we know that $f(I) \neq 0$ therefore $f(A) = 0$.

(iv) Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $P^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $K_k^{-1}PK_k = K_k^{-1}\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

If A has an inverse then $f(A) \neq 0$ since $1 = f(I) = f(A)f(A^{-1})$, in particular, $f(A)f(A^{-1}) = 1$. Using this for K_2 we have:

$f(P)^2 = f(P^2) = f(K_2^{-1}PK_2) = f(P)$ therefore $f(P) = 0, 1$, but since $f(P)$ has an inverse, $f(P) \neq 0$ so $f(P) = 1$