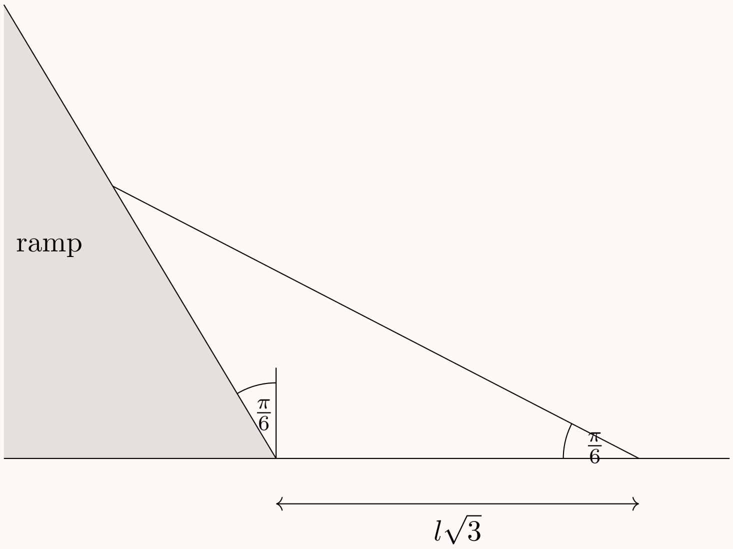
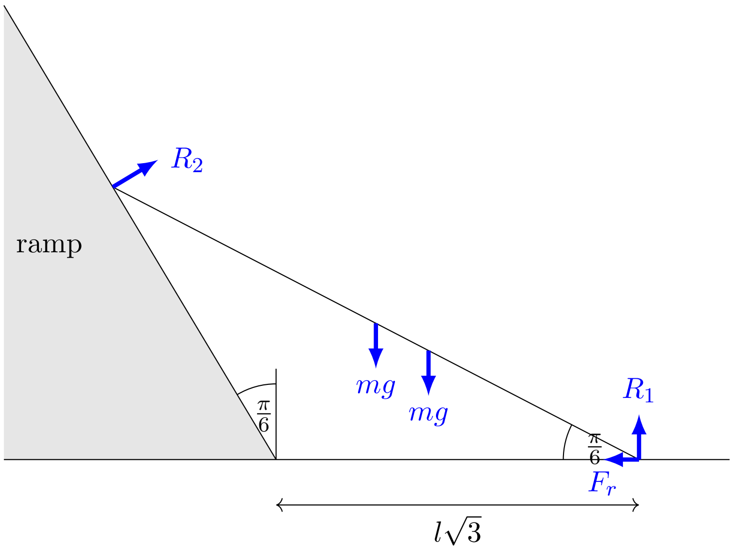
Question (1988 STEP III Q11)

A uniform ladder of length l and mass m rests with one end in contact with a smooth ramp inclined at an angle of $\pi/6$ to the vertical. The foot of the ladder rests, on horizontal ground, at a distance $l/\sqrt{3}$ from the foot of the ramp, and the coefficient of friction between the ladder and the ground is μ. The ladder is inclined at an angle $\pi/6$ to the horizontal, in the vertical plane containing a line of greatest slope of the ramp. A labourer of mass m intends to climb slowly to the top of the ladder.



(i) Find the value of μ if the ladder slips as soon as the labourer reaches the midpoint.

(ii) Find the minimum value of μ which will ensure that the labourer can reach the top of the ladder.



(i)

$$\text{N2}(\uparrow): \qquad R_1 + R_2\sin(\frac{\pi}{6}) - 2mg = 0$$

$$\text{N2}(\rightarrow): \quad R_2\cos(\frac{\pi}{6}) - F_r = 0$$
$$\overset{\curvearrowleft}{X}: \quad lmg\cos\tfrac{\pi}{6} - lR_2\cos\tfrac{\pi}{6} = 0$$

$$\Rightarrow \quad R_2 = mg$$
$$\Rightarrow \quad R_1 = 2mg - \frac{1}{2}mg$$
$$= \frac{3}{2}mg$$
$$\Rightarrow \quad \frac{\sqrt{3}}{2}mg - \mu\frac{3}{2}mg = 0$$
$$\Rightarrow \quad \mu = \frac{1}{\sqrt{3}}$$

(ii)

$$\text{N2}(\uparrow): \quad R_1 + R_2\sin(\frac{\pi}{6}) - 2mg = 0$$
$$\overset{\curvearrowleft}{X}: \quad \frac{1}{2}lmg\cos\tfrac{\pi}{6} + xmg\cos\tfrac{\pi}{6} - lR_2\cos\tfrac{\pi}{6} = 0$$

$$\Rightarrow \quad R_2 = mg(\frac{1}{2} + \frac{x}{l})$$
$$\Rightarrow \quad R_1 = 2mg - \frac{1}{2}mg(\frac{1}{2} + \frac{x}{l})$$
$$= (\frac{7}{4} - \frac{x}{2l})mg$$
$$\geq \frac{5}{4}mg$$
$$\text{N2}(\rightarrow): \quad R_2\cos(\frac{\pi}{6}) - \mu R_1 \leq 0$$
$$\Rightarrow \quad \frac{\sqrt{3}}{2}mg - \mu\frac{5}{4}mg \leq 0$$
$$\Rightarrow \quad \mu \geq \frac{2\sqrt{3}}{5}$$

Question (1989 STEP I Q13)

A uniform ladder of mass M rests with its upper end against a smooth vertical wall, and with its lower end on a rough slope which rises upwards towards the wall and makes an angle of ϕ with the horizontal. The acute angle between the ladder and the wall is θ. If the ladder is in equilibrium, show that N and F, the normal reaction and frictional force at the foot of the ladder are given by

$$N = Mg\left(\cos\phi - \frac{\tan\theta\sin\phi}{2}\right),$$

$$F = Mg\left(\sin\phi + \frac{\tan\theta\cos\phi}{2}\right).$$

If the coefficient of friction between the ladder and the slope is 2, and $\phi = 45°$, what is the largest value of θ for which the ladder can rest in equilibrium?

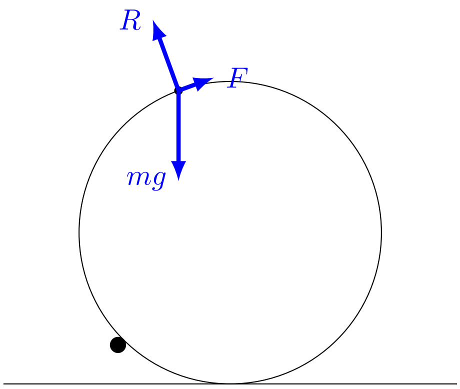
$\text{N2}(\uparrow): 0 = R\cos\phi + F\sin\phi - Mg$
$\text{N2}(\rightarrow): 0 = R_1 - F\cos\phi + R\sin\phi$
$\Rightarrow \frac{1}{2}\tan\theta Mg = F\cos\phi - R\sin\phi$
$Mg = F\sin\phi + R\cos\phi$
$\Rightarrow F = Mg\left(\sin\phi + \frac{1}{2}\tan\theta\cos\phi\right)$
$N = Mg\left(\cos\phi - \frac{1}{2}\tan\theta\sin\phi\right)$

If $\mu = 2$ and $\phi = 45°$, we must have $F \leq 2N$, so:

$$\begin{aligned}
& Mg\left(\sin\phi + \frac{1}{2}\tan\theta\cos\phi\right) \leq 2Mg\left(\cos\phi - \frac{1}{2}\tan\theta\sin\phi\right) \\
\Rightarrow \quad & 1 + \frac{1}{2}\tan\theta \leq 2 - \tan\theta \\
\Rightarrow \quad & \frac{3}{2}\tan\theta \leq 1 \\
\Rightarrow \quad & \tan\theta \leq \frac{2}{3} \\
\Rightarrow \quad & \theta \leq \tan^{-1}\frac{2}{3}
\end{aligned}$$

Question (1990 STEP I Q13)

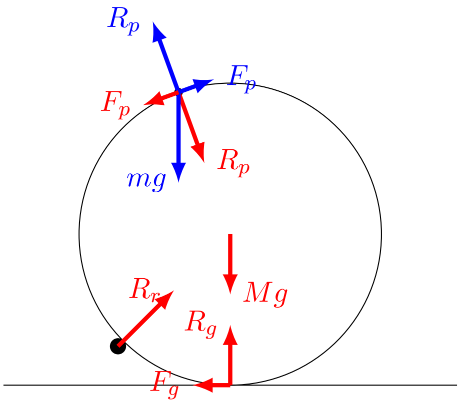
A rough circular cylinder of mass M and radius a rests on a rough horizontal plane. The curved surface of the cylinder is in contact with a smooth rail, parallel to the axis of the cylinder, which touches the cylinder at a height $a/2$ above the plane. Initially the cylinder is held at rest. A particle of mass m rests in equilibrium on the cylinder, and the normal reaction of the cylinder on the particle makes an angle of θ with the upward vertical. The particle is on the same side of the centre of the cylinder as the rail. The coefficient of friction between the cylinder and the particle and between the cylinder and the plane are both μ. Obtain the condition on θ for the particle to rest in equilibrium. Show that, if the cylinder is released, equilibrium of both particle and cylinder is possible provided another inequality involving μ and θ (which should be found explicitly) is satisfied. Determine the largest possible value of θ for equilibrium, if $m = 7M$ and $\mu = 0.75$.



$$\begin{aligned} \text{N2}(\nwarrow): \quad & R - mg\cos\theta = 0 \\ \text{N2}(\rightarrow): \quad & -R\sin\theta + F\cos\theta = 0 \\ \Rightarrow \quad & F = \tan\theta R \\ & F \leq \mu R \\ \Rightarrow \quad & \tan\theta R \leq \mu R \\ \Rightarrow \quad & \tan\theta \leq \mu \end{aligned}$$

(Notice also $F = mg\sin\theta$)

Once everything is released, we have the following situation. (Red forces act on the cylinder, blue forces on the particle).



$$\text{N2}(\uparrow): \quad 0 = R_g - Mg - \underbrace{mg}_{R_p \text{ and } F_p} + \frac{1}{\sqrt{2}}R_r$$

$$\text{N2}(\rightarrow): \quad 0 = \frac{1}{\sqrt{2}}R_r - F_g$$

$$\overset{\curvearrowleft}{O}: \quad 0 = aF_p - aF_g$$

$$\Rightarrow \quad F_g = mg\sin\theta$$

$$\Rightarrow \quad R_r = \sqrt{2}mg\sin\theta$$

$$\Rightarrow \quad R_g = (M+m)g + mg\sin\theta$$

$$F_g \leq \mu R_g$$

$$\Rightarrow \quad mg\sin\theta \leq \mu(M + m(1+\sin\theta))g$$

$$\Rightarrow \quad \mu \geq \frac{m\sin\theta}{M + m(1+\sin\theta)}$$

If $m = 7M$ and $\mu = \frac{3}{4}$ we have:

$$\tan\theta \leq \frac{3}{4}$$

$$3(M + 7M(1+\sin\theta)) \geq 4 \cdot 7M\sin\theta$$

$$\Rightarrow \quad 10 + 7\sin\theta \geq 28\sin\theta$$

$$\Rightarrow \quad 10 \geq 21\sin\theta$$

$$\Rightarrow \quad \sin\theta \leq \frac{10}{21}$$

If $\tan\theta = \frac{3}{4}, \sin\theta = \frac{3}{5} > \frac{10}{21}$, so the critical bound is $\sin\theta \leq \frac{10}{21}$, ie $\theta \leq \sin^{-1}\frac{10}{21} \approx 30°$

Question (1990 STEP III Q14)

The edges OA, OB, OC of a rigid cube are taken as coordinate axes and O', A', B', C' are the vertices diagonally opposite O, A, B, C, respectively. The four forces acting on the cube are

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \text{ at } O\,(0,0,0), \quad \begin{pmatrix}\lambda\\0\\1\end{pmatrix} \text{ at } O'\,(a,a,a), \quad \begin{pmatrix}-1\\0\\2\end{pmatrix} \text{ at } B\,(0,a,0), \quad \text{and} \quad \begin{pmatrix}1\\\mu\\\nu\end{pmatrix} \text{ at } B'\,(a,0,a).$$

The moment of the system about O is zero: find λ, μ and ν.

(i) Given that $\alpha = \beta = \gamma = 0$, find the system consisting of a single force at B together with a couple which is equivalent to the given system.

(ii) Given that $\alpha = 2, \beta = 3$ and $\gamma = 2$, find the equation of the locus about each point of which the moment of the system is zero. Find the number of units of work done on the cube when it moves (without rotation) a distance in the direction of this line under the action of the given forces only.

$$\begin{aligned}
\mathbf{M} &= \begin{pmatrix}\lambda\\0\\1\end{pmatrix} \times \begin{pmatrix}a\\a\\a\end{pmatrix} + \begin{pmatrix}-1\\0\\2\end{pmatrix} \times \begin{pmatrix}0\\a\\0\end{pmatrix} + \begin{pmatrix}1\\\mu\\\nu\end{pmatrix} \times \begin{pmatrix}a\\0\\a\end{pmatrix} \\
&= \begin{pmatrix}-a\\-a(\lambda-1)\\\lambda a\end{pmatrix} + \begin{pmatrix}-2a\\0\\-a\end{pmatrix} + \begin{pmatrix}\mu a\\-a(1-\nu)\\-a\mu\end{pmatrix} \\
&= a\begin{pmatrix}\mu-3\\\nu-\lambda\\\lambda-1-\mu\end{pmatrix} \\
\Rightarrow \qquad & \mu = 3, \lambda = 4, \nu = 4
\end{aligned}$$

(i) To find the force we add all vectors:

$$\begin{aligned}
\mathbf{F} &= \begin{pmatrix}\lambda\\0\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\2\end{pmatrix} + \begin{pmatrix}1\\\mu\\\nu\end{pmatrix} \\
&= \begin{pmatrix}4\\0\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\2\end{pmatrix} + \begin{pmatrix}1\\3\\4\end{pmatrix} \\
&= \begin{pmatrix}4\\3\\7\end{pmatrix}
\end{aligned}$$

Since the moment about O is 0, we have the moment about B is:

$$\mathbf{M} = \begin{pmatrix}0\\a\\0\end{pmatrix} \times \begin{pmatrix}4\\3\\7\end{pmatrix}$$

$$= \begin{pmatrix} 7a \\ 0 \\ -4a \end{pmatrix}$$

(ii)

$$\mathbf{0} = \mathbf{r} \times \begin{pmatrix} 4+2 \\ 3+3 \\ 7+2 \end{pmatrix}$$
$$= \mathbf{r} \times \begin{pmatrix} 6 \\ 6 \\ 9 \end{pmatrix}$$

Therefore $\mathbf{r} = t \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ (ie a line)

$$\text{Work done} = \text{Force} \cdot \text{distance}$$

Since they are parallel, it's just the magnitude of the force, which is $3\sqrt{2^2+2^2+3^2} = 3\sqrt{17}$

Question (1991 STEP I Q12)

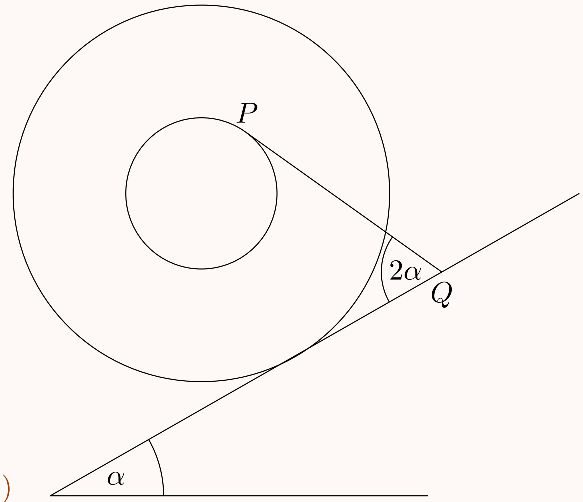
xunit=0.8cm,yunit=0.8cm,algebraic=true,dotstyle=o,dotsize=3pt 0,linewidth=0.5pt,arrowsize=3pt 2,arrowinset=0.25 (-2.4,-1.16)(12.46,5.7) (0,0)(6,4) (10,0)(5,5) [tl](5.08,5.53)D [tl](5.31,4.3)B [tl](3.39,2.99)$2l$ [fill-color=black,fillstyle=solid,opacity=0]0.00.58800260354756751.23*cos(t)+0—1.23*sin(t)+0(0,0) [tl](0.67,0.39)α [fill-color=black,fillstyle=solid,opacity=0]2.3561944901923453.1415926535897931.23*cos(t)+10—1.23*sin(t)+0 [tl](9.09,0.56)β -¿(8,2)(8,1.3) [tl](7.64,1.31)Mg [tl](9.17,1.38)x [tl](7.27,3.32)x [tl](-0.29,-0.18)A [tl](10.15,-0.2)C (-2,0)(12,0)

The above diagram illustrates a makeshift stepladder, made from two equal light planks AB and CD, each of length $2l$. The plank AB is smoothly hinged to the ground at A and makes an angle of α with the horizontal. The other plank CD has its bottom end C resting on the same horizontal ground and makes an angle β with the horizontal. It is pivoted smoothly to B at a point distance $2x$ from C. The coefficient of friction between CD and the ground is μ. A painter of mass M stands on CD, half between C and B. Show that, for equilibrium to be possible,

$$\mu \geqslant \frac{\cot\alpha \cot\beta}{2\cot\alpha + \cot\beta}.$$

Suppose now that B coincides with D. Show that, as α varies, the maximum distance from A at which the painter will be standing is

$$l\sqrt{\frac{1+81\mu^2}{1+9\mu^2}}.$$



Question (1991 STEP III Q11)

A uniform circular cylinder of radius $2a$ with a groove of radius a cut in its central cross-section has mass M. It rests, as shown in the diagram, on a rough plane inclined at an acute angle α to the horizontal. It is supported by a light inextensible string would round the groove and attached to the cylinder at one end. The other end of the string is attached to the plane at Q, the free part of the string, PQ, making an angle 2α with the inclined plane. The coefficient of friction at the contact between the cylinder and the plane is μ. Show that $\mu \geqslant \frac{1}{3}\tan\alpha$. The string PQ is now detached from the plane and the end Q is fastened to a particle of mass $3M$ which is placed on the plane, the position of the string remain unchanged. Given that $\tan\alpha = \frac{1}{2}$ and that the system remains in equilibrium, find the least value of the coefficient of friction between the particle and the plane.

None

Question (1992 STEP I Q12)

The diagram shows a crude step-ladder constructed by smoothly hinging-together two light ladders AB and AC, each of length l, at A. A uniform rod of wood, of mass m, is pin-jointed to X on AB and to Y on AC, where $AX = \frac{3}{4}l = AY$. The angle $\angle XAY$ is 2θ.

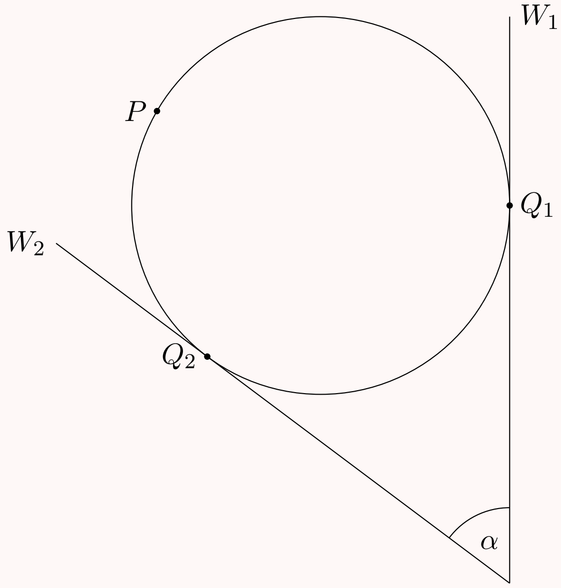
xunit=1.0cm,yunit=1.0cm,algebraic=true,dimen=middle,dotstyle=o,dotsize=3pt 0,linewidth=0.5pt,arrowsize=3pt 2,arrowinset=0.25 (-4.3,-1.22)(4.6,6) (-4,0)(4,0) (-2,0)(0,5) (0,5)(2,0) (-1.21,1.97)(1.21,1.97) -1.9513027039072617-1.1902899496825321.2*cos(t)+0—1.2*sin(t)+5 [tl](-0.2,4.26)2θ [tl](-0.1,5.5)A [tl](-1.8,2.1)X [tl](1.5,2.1)Y [tl](-2.36,-0.1)B [tl](2.02,-0.1)C

The rod XY will break if the tension in it exceeds T. The step-ladder stands on rough horizontal ground (coefficient of friction μ). Given that $\tan\theta > \mu$, find how large a mass M can safely be placed at A and show that if

$$\tan\theta > \frac{6T}{mg} + 4\mu$$

the step-ladder will fail under its own weight.

[You may assume that friction is limiting at the moment of collapse.]



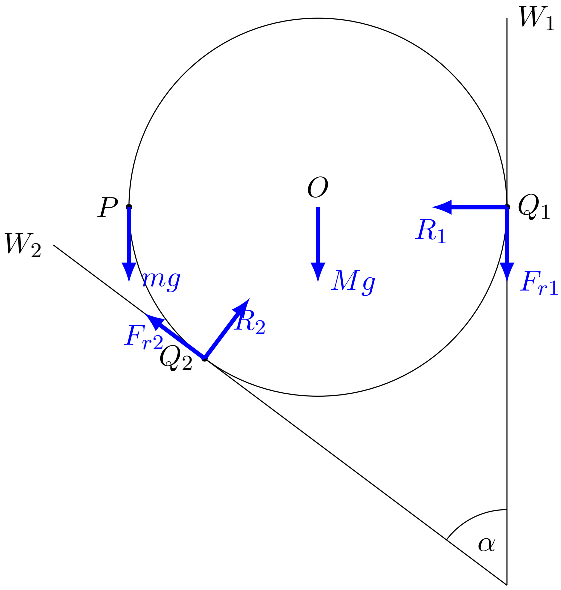
Question (1993 STEP II Q12)

A uniform sphere of mass M and radius r rests between a vertical wall W_1 and an inclined plane W_2 that meets W_1 at an angle α. Q_1 and Q_2 are the points of contact of the sphere with W_1 and W_2 resectively, as shown in the diagram. A particle of mass m is attached to the sphere at P, where PQ_1 is a diameter, and the system is released. The sphere is on the point of slipping at Q_1 and at Q_2. Show that if the coefficients of friction between the sphere and W_1 and W_2 are μ_1 and μ_2 respectively, then

$$m = \frac{\mu_2 + \mu_1 \cos\alpha - \mu_1\mu_2 \sin\alpha}{(2\mu_1\mu_2 + 1)\sin\alpha + (\mu_2 - 2\mu_1)\cos\alpha - \mu_2} M.$$

If the sphere is on the point of rolling about Q_2 instead of slipping, show that

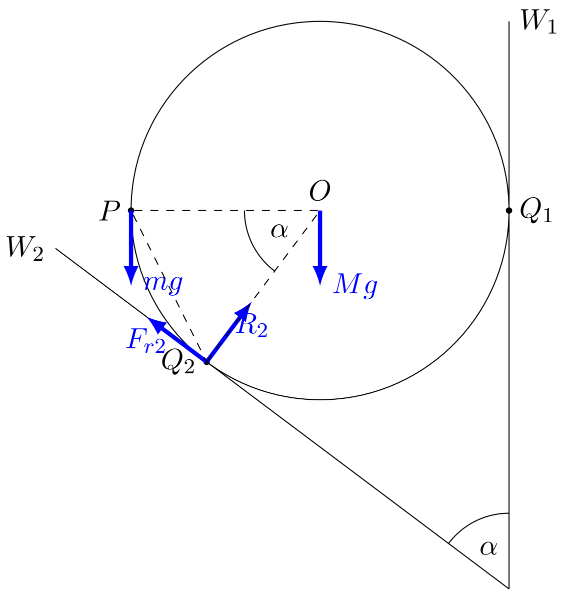
$$m = \frac{M}{\sec\alpha - 1}.$$



Since the sphere is on the point of slipping at both Q_1 and Q_2, $F_{r1} = \mu_1 R_1$ and $F_{r2} = \mu_2 R_2$

$$\begin{aligned}
\text{N2}(\uparrow): \quad & -mg - Mg - \mu_1 R_1 + R_2 \sin\alpha + \mu_2 R_2 \cos\alpha = 0 \\
\text{N2}(\rightarrow): \quad & -R_1 + R_2 \cos\alpha - \mu_2 R_2 \sin\alpha = 0 \\
\Rightarrow \quad & R_2 \cos\alpha - \mu_2 R_2 \sin\alpha = R_1 \\
\overset{\curvearrowleft}{O}: \quad & mg - \mu_1 R_1 - \mu_2 R_2 = 0 \\
\Rightarrow \quad & \mu_1 R_2 (\cos\alpha - \mu_2 \sin\alpha) - \mu_2 R_2 = -mg \\
& \mu_1(R_2 \cos\alpha - \mu_2 R_2 \sin\alpha) + R_2 \sin\alpha + \\
& \mu_2 R_2 \cos\alpha - \mu_1 R_2 (\cos\alpha - \mu_2 \sin\alpha) - \mu_2 R_2 = Mg \\
\Rightarrow \quad & \frac{\mu_2 + \mu_1 (\cos\alpha - \mu_2 \sin\alpha)}{\mu_1(\cos\alpha - \mu_2 \sin\alpha) + \sin\alpha + \mu_2 \cos\alpha - \mu_1 (\cos\alpha - \mu_2 \sin\alpha) - \mu_2} = \frac{m}{M} \\
& \frac{\mu_2 + \mu_1 \cos\alpha - \mu_1 \mu_2 \sin\alpha}{\cos\alpha(-2\mu_1 + \mu_2) + \sin\alpha(1 + 2\mu_1\mu_2) - \mu_2} = \frac{m}{M}
\end{aligned}$$

If instead the sphere is about to roll about Q_2, then the forces at Q_1 will be 0, we can then take moments about Q_2.



Looking at perpendicular distances from Q_2 to O and P we have $r\cos\alpha$ and $r(1-\cos\alpha)$

$$\begin{aligned}
\overset{\curvearrowleft}{Q_2}: \quad & mg(1-\cos\alpha) - Mg\cos\alpha = 0 \\
\Rightarrow \quad & \frac{1}{\sec\alpha - 1} = \frac{m}{M}
\end{aligned}$$

Question (1996 STEP III Q10)

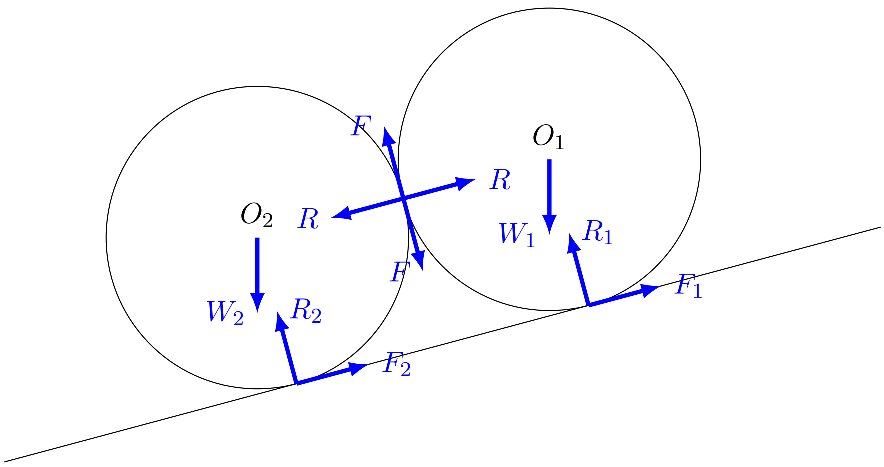
Two rough solid circular cylinders, of equal radius and length and of uniform density, lie side by side on a rough plane inclined at an angle α to the horizontal, where $0 < \alpha < \pi/2$. Their axes are horizontal and they touch along their entire length. The weight of the upper cylinder is W_1 and the coefficient of friction between it and the plane is μ_1. The corresponding quantities for the lower cylinder are W_2 and μ_2 respectively and the coefficient of friction between the two cylinders is μ. Show that for equilibrium to be possible:

(i) $W_1 \geq W_2$;

(ii) $\mu \geqslant \dfrac{W_1 + W_2}{W_1 - W_2}$;

(iii) $\mu_1 \geqslant \left(\dfrac{2W_1 \cot\alpha}{W_1 + W_2} - 1\right)^{-1}$.

Find the similar inequality to **(iii)** for μ_2.



(i)

$\text{N2}(\swarrow, 2) : 0 = R + W_2 \sin\alpha - F$
$\text{N2}(\swarrow, 1) : 0 = W_1 \sin\alpha - F - R$
$\Rightarrow W_1 \sin\alpha - R = W_2 \sin\alpha + R$
$\Rightarrow W_1 \geq W_2$

(ii)

$$(1) + (2) \Rightarrow \qquad F = \frac{1}{2}\sin\alpha(W_1 + W_2)$$
$$(1) - (2) \Rightarrow \qquad R = \frac{1}{2}\sin\alpha(W_1 - W_2)$$

$$\Rightarrow \quad \frac{F}{R} = \frac{W_1 + W_2}{W_1 - W_2}$$

$$\underbrace{\Rightarrow}_{F \leq \mu R} \quad \mu \geq \frac{W_1 + W_2}{W_1 - W_2}$$

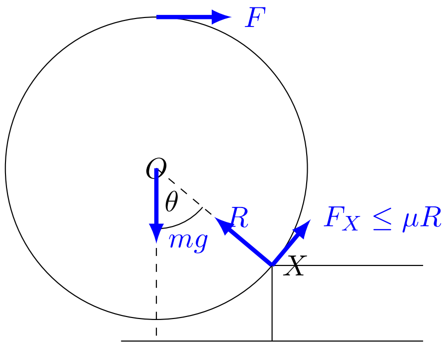
(iii)

$$\begin{aligned}
\text{N2}(\nwarrow, 1): \quad && 0 &= F + R_1 - W_1 \cos\alpha \\
\Rightarrow && R_1 &= W_1 \cos\alpha - F \\
&& &= W_1 \cos\alpha - \frac{1}{2}\sin\alpha(W_1 + W_2) \\
\Rightarrow && \frac{R_1}{F_1} &= \frac{R_1}{F} \\
&& &= \frac{W_1 \cos\alpha - \frac{1}{2}\sin\alpha(W_1 + W_2)}{\frac{1}{2}\sin\alpha(W_1 + W_2)} \\
&& &= \frac{2W_1 \cot\alpha}{W_1 + W_2} - 1 \\
\Rightarrow && \mu_1 &\geq \left(\frac{2W_1 \cot\alpha}{W_1 + W_2} - 1\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
\text{N2}(\nwarrow, 2): \quad && 0 &= -F + R_2 - W_2 \cos\alpha \\
\Rightarrow && R_2 &= W_2 \cos\alpha + F \\
&& &= W_2 \cos\alpha + \frac{1}{2}\sin\alpha(W_1 + W_2) \\
\Rightarrow && \frac{R_2}{F_2} &= \frac{R_2}{F} \\
&& &= \frac{W_2 \cos\alpha + \frac{1}{2}\sin\alpha(W_1 + W_2)}{\frac{1}{2}\sin\alpha(W_1 + W_2)} \\
&& &= \frac{2W_2 \cot\alpha}{W_1 + W_2} + 1 \\
\Rightarrow && \mu_2 &\geq \left(\frac{2W_1 \cot\alpha}{W_1 + W_2} + 1\right)^{-1}
\end{aligned}$$

Question (1997 STEP II Q9)

A uniform solid sphere of diameter d and mass m is drawn very slowly and without slipping from horizontal ground onto a step of height $d/4$ by a horizontal force which is always applied to the highest point of the sphere and is always perpendicular to the vertical plane which forms the face of the step. Find the maximum horizontal force throughout the movement, and prove that the coefficient of friction between the sphere and the edge of the step must exceed $1/\sqrt{3}$.



The ball is on the ground when $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$ and ball will make it over the step when $\theta = 0°$. It is also worth emphasising we are moving *very slowly*, so we can treat the system as static at any given point.

$$\begin{aligned}
\overset{\curvearrowleft}{X}: \quad & mg\frac{d}{2}\sin\theta - F\frac{d}{2}(1+\cos\theta) = 0 \\
\Rightarrow \quad & \frac{mg\sin\theta}{1+\cos\theta} = F \\
\Rightarrow \quad & mg\tan\frac{\theta}{2} = F
\end{aligned}$$

Therefore F is maximised when $\theta = 60°$, ie $F_{max} = \frac{mg}{\sqrt{3}}$

$$\begin{aligned}
\text{N2}(\parallel OX): \quad & mg\cos\theta - R + F\sin\theta = 0 \\
\Rightarrow \quad & mg\cos\theta - R + \frac{mg\sin\theta}{1+\cos\theta}\sin\theta = 0 \\
\Rightarrow \quad & mg = R
\end{aligned}$$

$$\begin{aligned}
\text{N2}(\perp OX): \quad & F_X - mg\sin\theta + F\cos\theta = 0 \\
\Rightarrow \quad & mg\sin\theta - \frac{mg\sin\theta}{1+\cos\theta}\cos\theta = F_X \\
\Rightarrow \quad & \frac{mg\sin\theta}{1+\cos\theta} = F_X
\end{aligned}$$

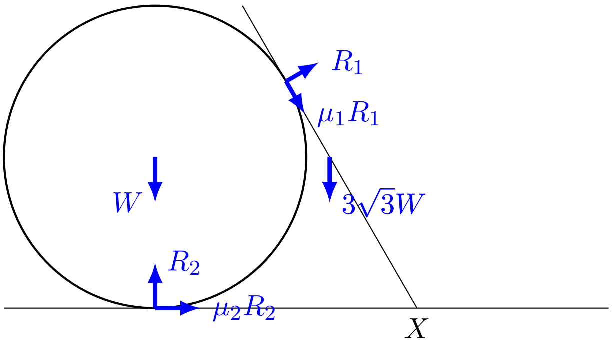
(We could also see this taking moments about O)

Therefore since $F_X \leq \mu R$, $\dfrac{mg\sin\theta}{1+\cos\theta} \leq \mu mg \Rightarrow \mu \geq \tan\dfrac{\theta}{2}$ which is maximised at $\theta = 60°$ and implies $\mu \geq \frac{1}{\sqrt{3}}$

Question (2000 STEP III Q10)

A sphere of radius a and weight W rests on horizontal ground. A thin uniform beam of weight $3\sqrt{3}\,W$ and length $2a$ is freely hinged to the ground at X, which is a distance $\sqrt{3}\,a$ from the point of contact of the sphere with the ground. The beam rests on the sphere, lying in the same vertical plane as the centre of the sphere. The coefficients of friction between the beam and the sphere and between the sphere and the ground are μ_1 and μ_2 respectively.

Given that the sphere is on the point of slipping at its contacts with both the ground and the beam, find the values of μ_1 and μ_2.



The first important thing to observe is the angle at X is 60°.
Now we can start resolving:

$$\overset{\curvearrowleft}{X}: \quad 3\sqrt{3}W\cos 60^\circ a - R_1\sqrt{3}a = 0 \quad (1)$$

$$\overset{\curvearrowleft}{O}: \quad \mu_2 R_2 a - \mu_1 R_1 a = 0 \quad (2)$$

$$\text{N2}(\rightarrow): \quad \mu_2 R_2 + \mu_1 R_1 \cos 60^\circ - R_1 \cos 30^\circ = 0 \quad (3)$$

$$\text{N2}(\uparrow): \quad R_2 - W - \mu_1 R_1 \cos 30^\circ - R_1 \cos 60^\circ = 0 \quad (4)$$

$$\Rightarrow \quad \frac{3}{2}W = R_1 \quad ((5) \text{ from } (1))$$

$$\mu_1 R_1 = \mu_2 R_2 \quad (2)$$

$$\mu_1 R_1 \left(1 + \frac{1}{2}\right) - R_1 \frac{\sqrt{3}}{2} = 0 \quad ((3) \text{ and } (2))$$

$$\mu_1 = \frac{1}{\sqrt{3}}$$

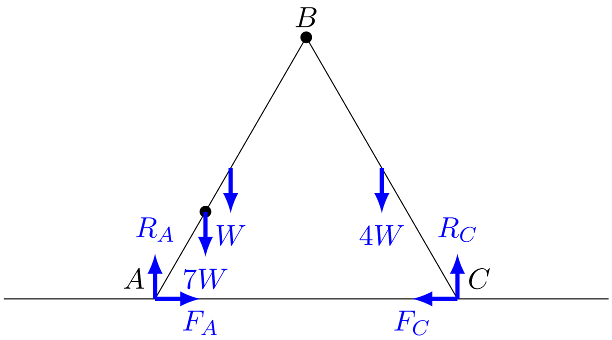
$$R_2 - W - \frac{1}{\sqrt{3}}\frac{3}{2}W\frac{\sqrt{3}}{2} - \frac{3}{2}W\frac{1}{2} = 0$$

$$\Rightarrow \quad R_2 = W\left(1 + \frac{3}{2}\right) \quad (6)$$

$$\Rightarrow \quad \mu_2 = \frac{\mu_1 R_1}{R_2} = \frac{1}{\sqrt{3}}\frac{3}{5} = \frac{\sqrt{3}}{5} \quad ((5) \text{ and } (6))$$

Question (2004 STEP I Q11)

Two uniform ladders AB and BC of equal length are hinged smoothly at B. The weight of AB is W and the weight of BC is $4W$. The ladders stand on rough horizontal ground with $\angle ABC = 60°$. The coefficient of friction between each ladder and the ground is μ. A decorator of weight $7W$ begins to climb the ladder AB slowly. When she has climbed up $\frac{1}{3}$ of the ladder, one of the ladders slips. Which ladder slips, and what is the value of μ?



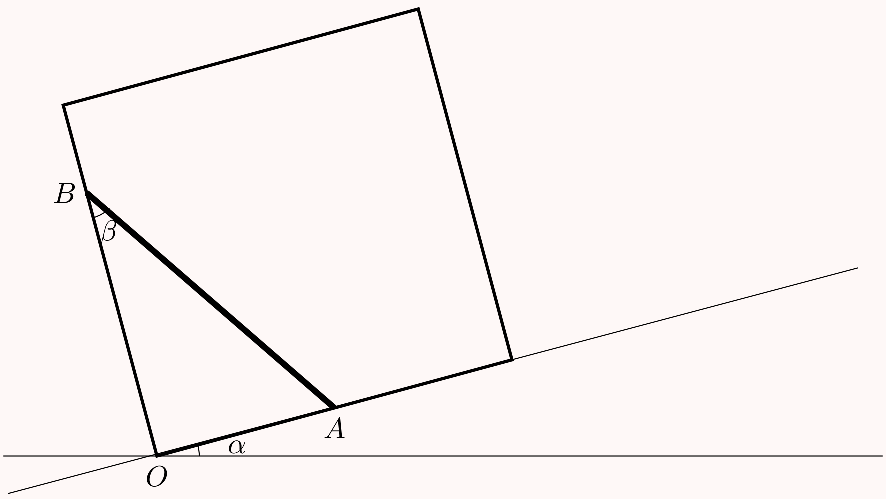
$$
\begin{aligned}
\text{N2}(\rightarrow): && F_A - F_C &= 0 \\
&& F_A &= F_C \\
\text{N2}(\uparrow): && R_A + R_C - 7W - W - 4W &= 0 \\
&& R_A + R_C &= 12W \\
\overset{\frown}{A}: && \frac{1}{6}7W + \frac{1}{4}W + \frac{3}{4}4W - R_C &= 0 \\
\Rightarrow && R_C &= \frac{53}{12}W \\
\Rightarrow && R_A = 12W - \frac{53}{12}W &= \frac{91}{12}W \\
\overset{\frown}{B}(AB): && \frac{1}{2}W + \frac{2}{3}7W - R_A + \sqrt{3}F_A &= 0 \\
\Rightarrow && F_A = \frac{1}{\sqrt{3}}\left(\frac{91}{12} - \frac{1}{2} - \frac{14}{3}\right)W &= \frac{29}{12\sqrt{3}}W
\end{aligned}
$$

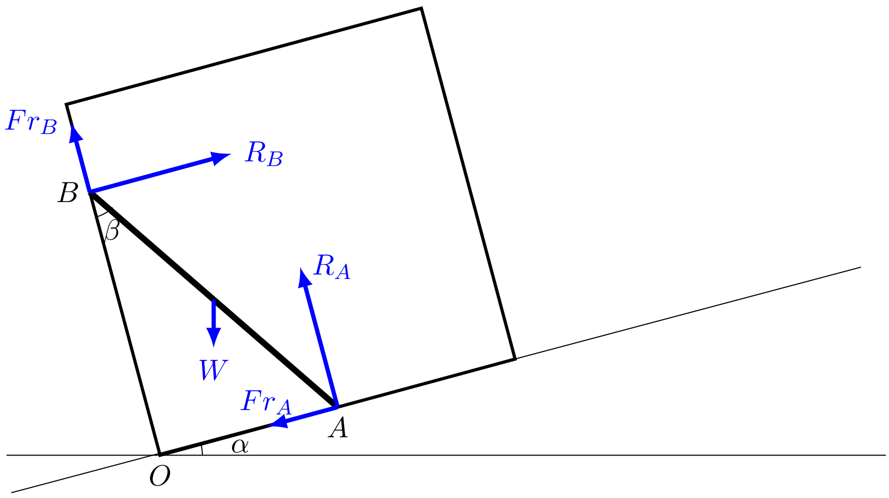
We know that the system is about to slip, so equality holds in one of $F_A \leq \mu R_A$ or $F_C \leq \mu R_C$. Since $F_A = F_C$, we know it must occur for whichever of μR_A and μR_C is smaller. Since R_C is much smaller, this must be the ladder about to slip BC and

$$\mu = \frac{F_C}{R_C} = \frac{\frac{29}{12\sqrt{3}}W}{\frac{53}{12}W} = \boxed{\frac{29}{53\sqrt{3}}}$$

Question (2017 STEP I Q11)

A plane makes an acute angle α with the horizontal. A box in the shape of a cube is fixed onto the plane in such a way that four of its edges are horizontal and two of its sides are vertical. A uniform rod of length $2L$ and weight W rests with its lower end at A on the bottom of the box and its upper end at B on a side of the box, as shown in the diagram below. The vertical plane containing the rod is parallel to the vertical sides of the box and cuts the lowest edge of the box at O. The rod makes an acute angle β with the side of the box at B. The coefficients of friction between the rod and the box at the two points of contact are both $\tan\gamma$, where $0 < \gamma < \frac{1}{2}\pi$. The rod is in limiting equilibrium, with the end at A on the point of slipping in the direction away from O and the end at B on the point of slipping towards O. Given that $\alpha < \beta$, show that $\beta = \alpha + 2\gamma$. [**Hint**: You may find it helpful to take moments about the midpoint of the rod.]





Since we're at limiting equilibrium and about to slip $Fr_B = \mu R_B$ and $Fr_A = \mu R_A$

$$\text{N2}(\parallel OB): \quad \mu R_B + R_A - W\cos\alpha = 0$$
$$\text{N2}(\parallel OA): \quad R_B - \mu R_A - W\sin\alpha = 0$$

$$\Rightarrow \quad \sin\alpha\,(\mu R_B + R_A) - \cos\alpha\,(R_B - \mu R_A) = 0$$

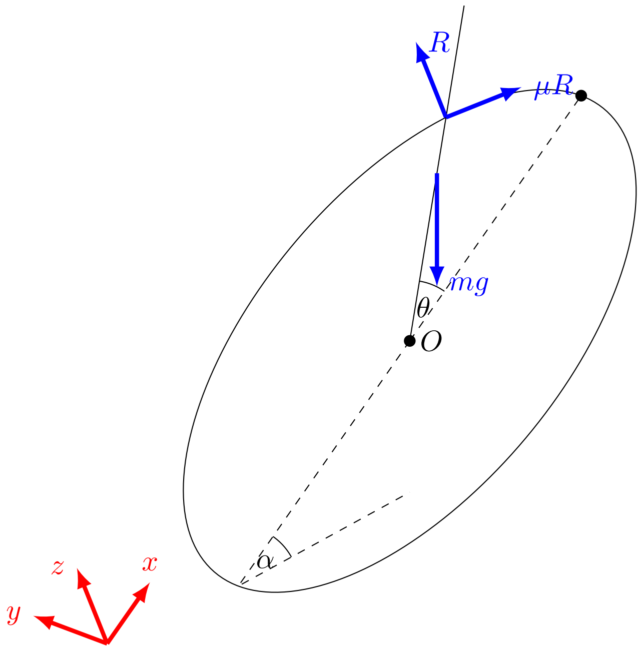
$$\begin{aligned}
\Leftrightarrow \qquad & R_A(\sin\alpha + \mu\cos\alpha) - R_B(\cos\alpha - \mu\sin\alpha) = 0 \\
\Rightarrow \qquad & \frac{\tan\alpha + \mu}{1 - \mu\tan\alpha} R_A = R_B \\
& \tan(\alpha+\gamma) R_A = R_B
\end{aligned}$$

$$\begin{aligned}
\overset{\curvearrowleft}{\text{midpoint}}: \qquad & R_A\sin\beta - \mu R_A\cos\beta - R_B\cos\beta - \mu R_B \sin\beta = 0 \\
\Rightarrow \qquad & \tan\beta - \mu - \tan(\alpha+\gamma) - \mu\tan(\alpha+\gamma)\tan\beta = 0 \\
\Rightarrow \qquad & \tan\beta\,(1 - \mu\tan(\alpha+\gamma)) - \mu - \tan(\alpha+\gamma) = 0 \\
\Rightarrow \qquad & \frac{\mu + \tan(\alpha+\gamma)}{1 - \mu\tan(\alpha+\gamma)} = \tan\beta \\
\Rightarrow \qquad & \tan(\alpha + 2\gamma) = \tan\beta
\end{aligned}$$

Since $\alpha < \beta$ and $\gamma < \frac{\pi}{4}$ we must have $\alpha + 2\gamma = \beta$

Question (1987 STEP II Q11)

A rough ring of radius a is fixed so that it lies in a plane inclined at an angle α to the horizontal. A uniform heavy rod of length $b(> a)$ has one end smoothly pivoted at the centre of the ring, so that the rod is free to move in any direction. It rests on the circumference of the ring, making an angle θ with the radius to the highest point on the circumference. Find the relation between α, θ and the coefficient of friction, μ, which must hold when the rod is in limiting equilibrium.



It is important to define clear coordinate axes, so let the x-axis point up the line of greatest slope of the ring. The z-axis perpendicular to the ring, and the y-axis perpendicular to both of these.

Our method is going to be to take moments about O to avoid worrying about the force at the pivot.

There are 3 forces we need to worry about:

- The mass of the rod
- The reaction where it meets the ring
- The friction at the ring

In our coordinate frame, the reaction will act in the z-direction, $\begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix}$, the friction force will act in the $x-y$ plane: $\begin{pmatrix} \mu R \sin\theta \\ -\mu R \cos\theta \\ 0 \end{pmatrix}$. We don't know the mass, but we know it will be acting "vertically", so $\cos\alpha$ of it will act in the z-axis and $\sin\alpha$ will act in the y-axis, ie it will act parallel to $\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}$.

When taking moments, we need to consider $\mathbf{r}$ the direction of the rod. This will be $\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$.

The moment of the weight will all be parallel to $\mathbf{r} \times \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}$. Similarly the moments of the contact forces will be $\mathbf{r} \times \begin{pmatrix} \mu R \sin\theta \\ -\mu R \cos\theta \\ R \end{pmatrix}$. Since these moments sum to $\mathbf{0}$ as we are in equilibrium, these vectors must be parallel.

Therefore it is sufficient to check the vector triple product,

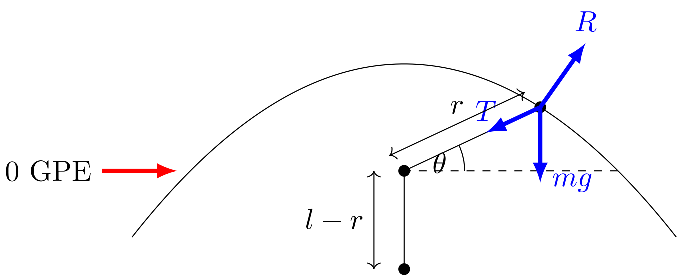
$$\begin{aligned} 0 &= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \cdot \left(\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} \times \begin{pmatrix} \mu \sin\theta \\ -\mu\cos\theta \\ 1 \end{pmatrix} \right) \\ &= \cos\theta(\mu\cos\theta\cos\alpha) - \sin\theta(\sin\alpha - \mu\sin\theta\cos\alpha) \\ &= \mu((\sin^2\theta + \cos^2\theta)\cos\alpha) - \sin\theta\sin\alpha \\ \Rightarrow \qquad \mu &= \tan\alpha\sin\theta \end{aligned}$$

Question (1987 STEP II Q12)

A long, inextensible string passes through a small fixed ring. One end of the string is attached to a particle of mass m, which hangs freely. The other end is attached to a bead also of mass m which is threaded on a smooth rigid wire fixed in the same vertical plane as the ring. The curve of the wire is such that the system can be in static equilibrium for all positions of the bead. The shortest distance between the wire and the ring is $d(>0)$. Using plane polar coordinates centred on the ring, find the equation of the curve. The bead is set in motion. Assuming that the string remains taut, show that the speed of the bead when it is a distance r from the ring is

$$\left(\frac{r}{2r-d}\right)^{\frac{1}{2}} v,$$

where v is the speed of the bead when $r = d$.



Assume the total length of the string is l. Then the total energy of the system (when nothing is moving) for a given θ is: $mg(r-l)+mgr\sin\theta$

Since for a point in static equilibrium, the derivative of this must be 0, this must be constant. So: $r(\sin\theta+1)=C \Rightarrow r=\frac{C}{1+\sin\theta}$

r will be smallest when $\sin\theta=1$, ie in polar coordinates, the equation should be $r=\frac{2d}{1+\sin\theta}$

Alternatively, by considering forces, the shape must be a parabola with the ring at the focus.

Considering the bead, it will have speed of $r\dot{\theta}$ tangentially, and $-\dot{r}$. The other particle will have speed $\dot{r}$.

Differentiating wrt to t

$$\begin{aligned} 0 &= \dot{r}(\sin\theta+1)+r\dot{\theta}\cos\theta \\ \Rightarrow \qquad \dot{\theta} &= \frac{-\dot{r}(1+\sin\theta)}{r\cos\theta} \\ &= \frac{-\dot{r}2d}{r^2\sqrt{1-\left(\frac{2d}{r}-1\right)^2}} \\ &= \frac{-2d\dot{r}}{r^2\sqrt{\frac{r^2-(2d-r)^2}{r^2}}} \\ &= \frac{-d\dot{r}}{r\sqrt{dr-d^2}} \end{aligned}$$

By conservation of energy (since GPE is constant throughout the system, KE must be constant):

$$\begin{aligned} \frac{1}{2}m(r^2\dot{\theta}^2+\dot{r}^2)+\frac{1}{2}m\dot{r}^2 &= \frac{1}{2}mv^2 \\ \Rightarrow \qquad v^2 &= r^2\dot{\theta}^2+2\dot{r}^2 \\ &= r^2\frac{d^2\dot{r}^2}{r^2(dr-d^2)}+2\dot{r}^2 \\ &= \dot{r}^2\left(\frac{d}{r-d}+2\right) \\ &= \dot{r}^2\left(\frac{2r-d}{r-d}\right) \\ \Rightarrow \qquad v &= \dot{r}\left(\frac{2r-d}{r-d}\right)^{\frac{1}{2}} \\ \Rightarrow \qquad \dot{r} &= \left(\frac{r-d}{2r-d}\right)^{\frac{1}{2}}v \end{aligned}$$

$$\begin{aligned}\Rightarrow \qquad u^2 &= r^2\dot{\theta}^2 + \dot{r}^2 \\ &= \dot{r}^2\left(\frac{d}{r-d}+1\right) \\ &= \left(\frac{r-d}{2r-d}\right)\left(\frac{r}{r-d}\right)v^2 \\ &= \left(\frac{d}{2r-d}\right)v^2 \\ \Rightarrow \qquad u &= \left(\frac{d}{2r-d}\right)^{\frac{1}{2}} v\end{aligned}$$

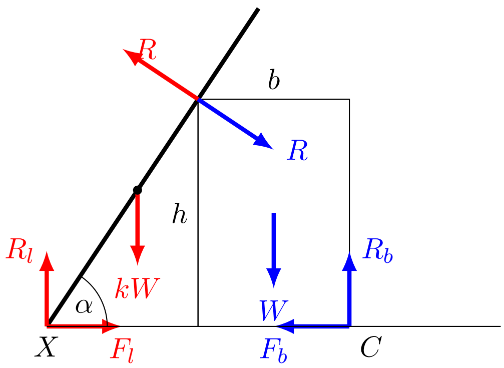
Question (2019 STEP I Q9)

A box has the shape of a uniform solid cuboid of height h and with a square base of side b, where $h > b$. It rests on rough horizontal ground. A light ladder has its foot on the ground and rests against one of the upper horizontal edges of the box, making an acute angle of α with the ground, where $h = b\tan\alpha$. The weight of the box is W. There is no friction at the contact between ladder and box. A painter of weight kW climbs the ladder slowly. Neither the base of the ladder nor the box slips, but the box starts to topple when the painter reaches height λh above the ground, where $\lambda < 1$. Show that:

(i) $R = k\lambda W\cos\alpha$, where R is the magnitude of the force exerted by the box on the ladder;

(ii) $2k\lambda\cos 2\alpha + 1 = 0$;

(iii) $\mu \geq \frac{\sin 2\alpha}{1-3\cos 2\alpha}$, where μ is the coefficient of friction between the box and the ground.



At the point we are about to topple, reaction and friction forces will be acting at C

(i)

$$\begin{aligned}\overset{\frown}{X}: \qquad kW\cdot\lambda h\cos\alpha - Rh &= 0 \\ \Rightarrow \qquad R &= k\lambda W\cos\alpha\end{aligned}$$

(ii)

$$\begin{aligned}
\overset{\frown}{C}: \quad && R\sin\alpha\cdot h - R\cos\alpha\cdot b - W\frac{b}{2} &= 0 \\
&& k\lambda W\cos\alpha\sin\alpha\cdot b\tan\alpha - k\lambda W\cos\alpha\cos\alpha\cdot h - W\frac{b}{2} &= 0 \\
&& k\lambda(\cos^2\alpha - \sin^2\alpha) + \frac{1}{2} &= 0 \\
\Rightarrow \quad && 2k\lambda\cos 2\alpha + 1 &= 0
\end{aligned}$$

(iii)

$$\begin{aligned}
\text{N2}(\uparrow): \quad && R_b - W - R\cos\alpha &= 0 \\
\Rightarrow \quad && R_b &= W + k\lambda W\cos^2\alpha \\
\text{N2}(\rightarrow): \quad && R\sin\alpha - F_b &= 0 \\
\Rightarrow \quad && F_b &= R\sin\alpha
\end{aligned}$$

$$\begin{aligned}
&& F_b &\leq \mu R \\
\Rightarrow \quad && k\lambda W\cos\alpha\sin\alpha &= \mu(W + k\lambda W\cos^2\alpha) \\
\Rightarrow \quad && \mu &\geq \frac{k\lambda\cos\alpha\sin\alpha}{1 + k\lambda\cos^2\alpha} \\
&& &= \frac{k\lambda\sin 2\alpha}{2 + 2k\lambda cos^2\alpha} \\
&& &= \frac{k\lambda\sin 2\alpha}{2 + k\lambda(\cos 2\alpha + 1)} \\
&& &= \frac{k\lambda\sin 2\alpha}{-4k\lambda\cos 2\alpha + k\lambda(\cos 2\alpha + 1)} \\
&& &= \frac{\sin 2\alpha}{1 - 3\cos 2\alpha}
\end{aligned}$$

Question (2025 STEP III Q10)

A plank AB of length L initially lies horizontally at rest along the x-axis on a flat surface, with A at the origin. Point C on the plank is such that AC has length sL, where $0 < s < 1$. End A is then raised vertically along the y-axis so that its height above the horizontal surface at time t is $h(t)$, while end B remains in contact with the flat surface and on the x-axis. The function $h(t)$ satisfies the differential equation

$$\frac{d^2h}{dt^2} = -\omega^2 h, \text{ with } h(0) = 0 \text{ and } \frac{dh}{dt} = \omega L \text{ at } t = 0$$

where ω is a positive constant. A particle P of mass m remains in contact with the plank at point C.

(i) Show that the x-coordinate of P is $sL\cos\omega t$, and find a similar expression for its y-coordinate.

(ii) Find expressions for the x- and y-components of the acceleration of the particle.

(iii) N and F are the upward normal and frictional components, respectively, of the force of the plank on the particle. Show that

$$N = mg(1 - k\sin\omega t)\cos\omega t$$

and that

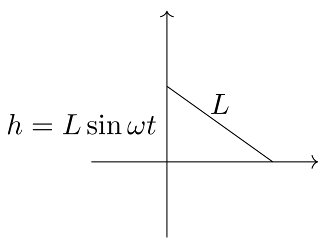
$$F = mgsk\frac{\omega^2}{g}\tan\omega t$$

where $k = \frac{L\omega^2}{g}$.

(iv) The coefficient of friction between the particle and the plank is $\tan\alpha$, where α is an acute angle. Show that the particle will not slip initially, provided $sk < \tan\alpha$. Show further that, in this case, the particle will slip

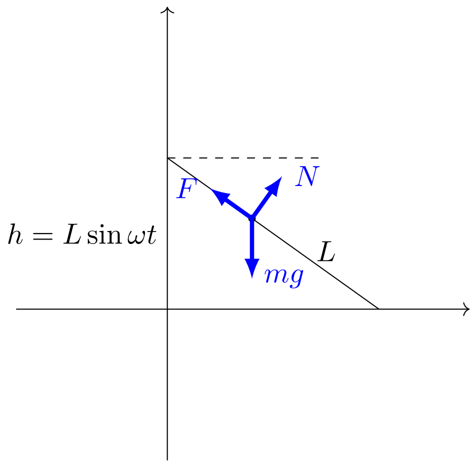
- while N is still positive,
- when the plank makes an angle less than α to the horizontal.

(i) Since we have $h'' + \omega^2 h = 0$ we must have that $h(t) = A\cos\omega t + B\sin\omega t$. The initial conditions tell us that $A = 0$ and $B = L$, so $h(t) = L\sin\omega t$.



Therefore we can see the angle at B is ωt and so P has y-coordinate $(1-s)L\sin\omega t$ and x-coordinate $sL\cos\omega t$

(ii) If the position is $\binom{sL\cos\omega t}{(1-s)L\sin\omega t}$ then the acceleration is $-\omega^2\binom{sL\cos\omega t}{(1-s)L\sin\omega t}$



(iii)

$$\begin{aligned}
\text{N2}(\rightarrow): \quad & -F\cos\omega t + N\sin\omega t = -m\omega^2 sL\cos\omega t \\
\text{N2}(\uparrow): \quad & -mg + F\sin\omega t + N\cos\omega t = -m\omega^2(1-s)L\sin\omega t \\
\Rightarrow \quad & \begin{pmatrix}\cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix}F \\ N\end{pmatrix} = \begin{pmatrix} m\omega^2 sL\cos\omega t \\ mg - m\omega^2(1-s)L\sin\omega t\end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix}F \\ N\end{pmatrix} = \begin{pmatrix}\cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix} m\omega^2 sL\cos\omega t \\ mg - m\omega^2(1-s)L\sin\omega t\end{pmatrix} \\
\Rightarrow \quad & N = m\omega^2 sL(-\sin\omega t\cos\omega t) + mg\cos\omega t - m\omega^2(1-s)L\sin\omega t\cos\omega \\
& \quad = mg\cos\omega t - m\omega^2 L\sin\omega t\cos\omega t \\
& \quad = mg\cos\omega t\left(1 - \frac{L\omega^2}{g}\sin\omega t\right) \\
& \quad = mg(1-k\sin\omega t)\cos\omega t \\
\Rightarrow \quad & F = m\omega^2 sL\cos^2\omega t + mg\sin\omega t - m\omega^2(1-s)L\sin^2\omega t \\
& \quad = m\omega^2 sL + mg\sin\omega t - m\omega^2 L\sin^2\omega t \\
& \quad = mg\frac{\omega^2 L}{g}s + mg(1 - \frac{\omega^2 L}{g}\sin\omega t)\sin\omega t \\
& \quad = mgsk + mg(1-k\sin\omega t)\cos\omega t\tan\omega t \\
& \quad = mgsk + N\tan\omega t
\end{aligned}$$

(iv) The particle will not slip if $F < \tan\alpha N$. When $t = 0$, $N = mg, F = mgsk$, but clearly $sk < \tan\alpha \Rightarrow mgsk = F < \tan\alpha mg = \tan\alpha N$.

The particle will slip when: $F > \tan\alpha N$, but we have $F = mgsk + N\tan\omega t$. Clearly when $\omega t = \alpha$ we have reached a point where $F > \tan\alpha N$. Therefore we must slip before we reach this point, ie at a point where the plank makes an angle of less than α to the horizontal. Notice also that N changes sign when $1 - k\sin\omega t = 0$, however, to do this N must become very small, smaller than $mgsk$, therefore we must slip before this point too. Since we slip before either condition occurs, we must be in a position when N is positive AND the plank still makes a shallow angle.