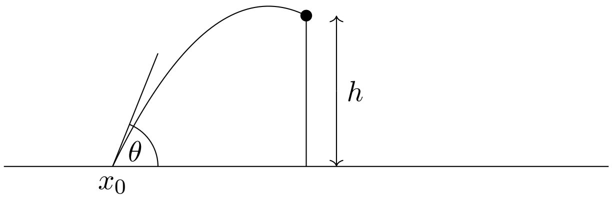
Question (1988 STEP I Q10)

A sniper at the top of a tree of height h is hit by a bullet fired from the undergrowth covering the horizontal ground below. The position and elevation of the gun which fired the shot are unknown, but it is known that the bullet left the gun with speed v. Show that it must have been fired from a point within a circle centred on the base of the tree and of radius $(v/g)\sqrt{v^2 - 2gh}$. [Neglect air resistance.]



The initial velocity is $\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$. The trajectory will be:

$\begin{pmatrix} x_0 + (v\cos\theta)t \\ (v\sin\theta)t - \frac{1}{2}gt^2 \end{pmatrix}$

we must have that for some time t, this is equal to $\begin{pmatrix} 0 \\ h \end{pmatrix}$

So $t = -\frac{x_0}{v\cos\theta}$ and so

$$\begin{aligned}
h &= (v\sin\theta)t - \frac{1}{2}gt^2 \\
&= -x_0\tan\theta - \frac{1}{2}g\frac{x_0^2}{v^2\cos^2\theta} \\
&= -x_0\tan\theta - \frac{g}{2v^2\cos^2\theta}x_0^2 \\
&= -x_0\tan\theta - \frac{g}{2v^2}\sec^2\theta x_0^2 \\
&= -x_0\tan\theta - \frac{g}{2v^2}(1+\tan^2\theta)x_0^2 \\
&= -\left(\frac{\sqrt{g}x_0}{\sqrt{2}v}\tan\theta + \frac{\sqrt{2}v}{2\sqrt{g}}\right)^2 + \frac{v^2}{2g} - \frac{g}{2v^2}x_0^2 \\
\Rightarrow \quad \frac{g}{2v^2}x_0^2 &= \frac{v^2}{2g} - h - \left(\frac{\sqrt{g}x_0}{\sqrt{2}v}\tan\theta + \frac{\sqrt{2}v}{2\sqrt{g}}\right)^2 \\
\Rightarrow \quad x_0^2 &= \frac{v^2(v^2-2gh)}{g^2} - K^2
\end{aligned}$$

Therefore $|x_0| \leq \frac{v}{g}\sqrt{v^2 - 2gh}$

Question (1989 STEP I Q11)

A shot-putter projects a shot at an angle θ above the horizontal, releasing it at height h above the level ground, with speed v. Show that the distance R travelled horizontally by the shot from its point of release until it strikes the ground is given by

$$R = \frac{v^2}{2g}\sin 2\theta\left(1+\sqrt{1+\frac{2hg}{v^2\sin^2\theta}}\right).$$

The shot-putter's style is such that currently $\theta = 45°$. Determine (with justification) whether a small decrease in θ will increase R. [Air resistance may be neglected.]

Notice that $u_x = v\cos\theta, u_y = v\sin\theta$.

We are interested in the time taken for the shot to hit the ground. $-h = u_y t - \frac{1}{2}gt^2$ since our distance will be $v\cos\theta\cdot t$.

Solving this quadratic for t we obtain:

$$\begin{aligned}
0 &= h + v\sin\theta\cdot t - \frac{1}{2}g\cdot t^2 \\
\Rightarrow \quad t_{\pm} &= \frac{-v\sin\theta \pm \sqrt{v^2\sin^2\theta + 2hg}}{-g} \\
\Rightarrow \quad t_{-} &= \frac{v\sin\theta + v\sin\theta\sqrt{1+\frac{2hg}{v^2\sin^2\theta}}}{g} \\
\Rightarrow \quad v\cos\theta t_{-} &= \frac{v^2}{g}\cos\theta\sin\theta\left(1+\sqrt{1+\frac{2hg}{v^2\sin^2\theta}}\right) \\
&= \frac{v^2}{2g}\sin 2\theta\left(1+\sqrt{1+\frac{2hg}{v^2\sin^2\theta}}\right)
\end{aligned}$$

Differentiating R wrt to θ at $\frac{\pi}{4}$ we obtain:

$$\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}\theta} &= \frac{v^2}{2g}\left(2\cos 2\theta + 2\cos 2\theta\sqrt{1+\frac{2hg}{v^2\sin^2\theta}} + \sin 2\theta\left(1+\frac{2hg}{v^2\sin^2\theta}\right)^{-\frac{1}{2}}\frac{1}{2}\frac{2hg}{v^2}(-2)\frac{\cos\theta}{\sin^3\theta}\right) \\
\left.\frac{\mathrm{d}R}{\mathrm{d}\theta}\right|_{\theta=\frac{\pi}{4}} &= \frac{v^2}{2g}\left(0+0-4\left(1+\frac{4hg}{v^2}\right)^{-\frac{1}{2}}\frac{hg}{v^2}\right) \\
&< 0
\end{aligned}$$

Therefore, since R is locally decreasing in θ he should reduce his angle of projection slightly.

Question (1990 STEP I Q11)

A shell of mass m is fired at elevation $\pi/3$ and speed v. Superman, of mass $2m$, catches the shell at the top of its flight, by gliding up behind it in the same horizontal direction with speed $3v$. As soon as Superman catches the shell, he instantaneously clasps it in his cloak, and immediately pushes it vertically downwards, without further changing its horizontal component of velocity, but giving it a downward vertical component of velocity of magnitude $3v/2$. Calculate the total time of flight of the shell in terms of v and g. Calculate also, to the nearest degree, the angle Superman's flight trajectory initially makes with the horizontal after releasing the shell, as he soars upwards like a bird. [Superman and the shell may be regarded as particles.]

The particle has initial velocity $\begin{pmatrix} v\cos\frac{\pi}{3} \\ v\sin\frac{\pi}{3} \end{pmatrix}$ and acceleration $\begin{pmatrix} 0 \\ -g \end{pmatrix}$. It will have zero vertical speed (ie be at the top of its trajectory) when $t = \frac{\sqrt{3}v}{2g}$.

Since $0 = v^2 - u^2 + 2as$ the height achieved will be $\frac{3v^2}{8g}$

At this point it will need to travel the same distance again, but this time the initial speed is $\frac{3v}{2}$ so:

$$\begin{aligned} \frac{3v^2}{8g} &= \frac{3v}{2}t + \frac{1}{2}gt^2 \\ \Rightarrow \quad 0 &= 4g^2t^2 + 12vgt - 3v^2 \\ \Rightarrow \quad t &= \left(\frac{-3+2\sqrt{3}}{2}\right)\frac{v}{g} \end{aligned}$$

Therefore the total time is:

$$\left(\frac{\sqrt{3}}{2} - \frac{3}{2} + \sqrt{3}\right)\frac{v}{g} = \frac{3\sqrt{3}-3}{2}\frac{v}{g}$$

$$\begin{aligned} COM(\uparrow): \quad 0 &= 2mv_y - m\frac{3}{2}v \\ \Rightarrow \quad v_y &= \frac{3}{4}v \\ COM(\rightarrow): \quad 3mV &= 2m(3v) + m\frac{v}{2} \\ \Rightarrow \quad V &= \frac{13}{6} \end{aligned}$$

Therefore superman is now travelling at a vector of $\begin{pmatrix} \frac{13}{6} \\ \frac{3}{4} \end{pmatrix} v$ ie an angle of $\tan^{-1}\frac{9}{26}$ to the horizontal, approximately $19°$

Question (1991 STEP II Q11)

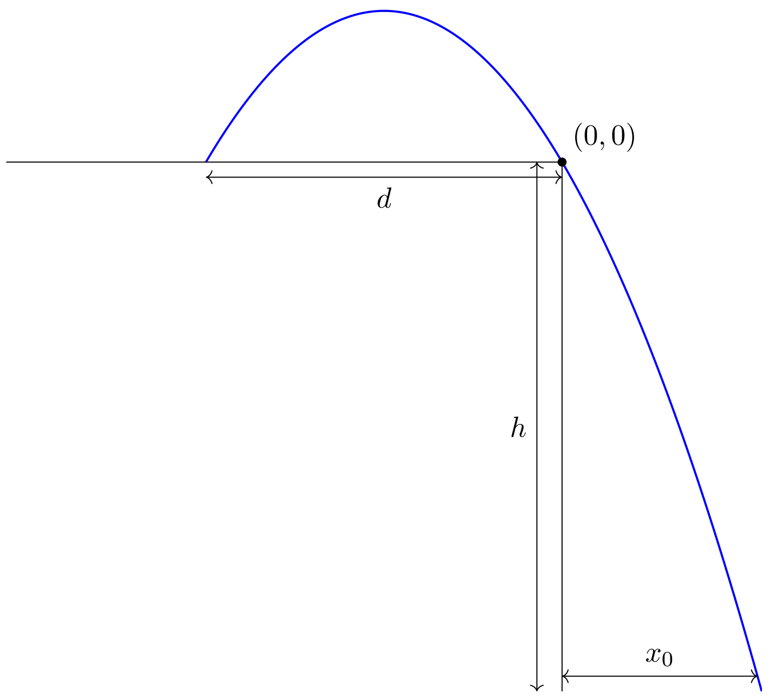
The Ruritanian army is supplied with shells which may explode at any time in flight but not before the shell reaches its maximum height. The effect of the explosion on any observer depends only on the distance between the exploding shell and the observer (and decreases with distance). Ruritanian guns fire the shells with fixed muzzle speed, and it is the policy of the gunners to fire the shell at an angle of elevation which minimises the possible damages to themselves (assuming the ground is level) - i.e. they aim so that the point on the descending trajectory that is nearest to them is as far away as possible. With that intention, they choose the angle of elevation that minimises the damage to themselves if the shell explodes at its maximum height. What angle do they choose?

Does the shell then get any nearer to the gunners during its descent?

Question (1992 STEP II Q11)

I am standing next to an ice-cream van at a distance d from the top of a vertical cliff of height h. It is not safe for me to go any nearer to the top of the cliff. My niece Padma is on the broad level beach at the foot of the cliff. I have just discovered that I have left my wallet with her, so I cannot buy her an ice-cream unless she can throw the wallet up to me. She can throw it at speed V, at any angle she chooses and from anywhere on the beach. Air resistance is negligible; so is Padma's height compared to that of the cliff. Show that she can throw the wallet to me if and only if

$$V^2 \geqslant g(2h + d).$$



Rather than considering Padma's throw, imagine a throw in reverse from me. As we can see from the diagram, it will need to pass through $(0, 0)$ to have minimal speed when it hits the ground, so possible throws are:

$$\begin{aligned}
0 &= u\sin\alpha t - \frac{1}{2}gt^2 \\
\Rightarrow \quad T &= \frac{2u\sin\alpha}{g} \\
d &= u\cos\alpha T \\
\Rightarrow \quad \frac{d}{u\cos\alpha} &= \frac{2u\sin\alpha}{g} \\
\Rightarrow \quad dg &= u^2\sin 2\alpha \\
v^2 &= u^2 + 2as \\
\Rightarrow \quad V_y^2 &= u^2\sin^2\alpha + 2gh \\
\Rightarrow \quad V^2 &= u^2\sin^2\alpha + 2gh + u^2\cos^2\theta \\
&= u^2 + 2gh \\
&= 2gh + \frac{dg}{\sin 2\alpha} \geq 2gh + dg = g(2h+d)
\end{aligned}$$

Question (1993 STEP I Q12)

In a clay pigeon shoot the target is launched vertically from ground level with speed v. At a time T later the competitor fires a rifle inclined at angle α to the horizontal. The competitor is also at ground level and is a distance l from the launcher. The speed of the bullet leaving the rifle is u. Show that, if the competitor scores a hit, then

$$l\sin\alpha - \left(vT - \tfrac{1}{2}gT^2\right)\cos\alpha = \frac{v-gT}{u}l.$$

Suppose now that $T = 0$. Show that if the competitor can hit the target before it hits the ground then $v < u$ and

$$\frac{2v\sqrt{u^2-v^2}}{g} > l.$$

None

Question (1994 STEP I Q9)

A cannon-ball is fired from a cannon at an initial speed u. After time t it has reached height h and is at a distance $\sqrt{x^2+h^2}$ from the cannon. Ignoring air resistance, show that

$$\tfrac{1}{4}g^2t^4 - (u^2 - gh)t^2 + h^2 + x^2 = 0.$$

Hence show that if $u^2 > 2gh$ then the horizontal range for a given height h and initial speed u is less than or equal to

$$\frac{u\sqrt{u^2-2gh}}{g}.$$

Show that there is always an angle of firing for which this value is attained.

Question (1994 STEP II Q11)

As part of a firework display a shell is fired vertically upwards with velocity v from a point on a level stretch of ground. When it reaches the top of its trajectory an explosion it splits into two equal fragments each travelling at speed u but (since momentum is conserved) in exactly opposite (not necessarily horizontal) directions. Show, neglecting air resistance, that the greatest possible distance between the points where the two fragments hit the ground is $2uv/g$ if $u \leqslant v$ and $(u^2+v^2)/g$ if $v \leqslant u$.

Since $v^2 - u^2 = 2as$ we have the initial height reached is $\frac{v^2}{2g}$.

At the point of explosion, the velocities are $\pm\binom{u\cos\theta}{u\sin\theta}$ where $0 \leq \theta < \frac{\pi}{2}$.
Looking vertically:

$$
\begin{aligned}
 & -\frac{v^2}{2g} = \pm u\sin\theta t - \frac{1}{2}gt^2 \\
\Rightarrow \quad & t = \frac{\mp u\sin\theta \pm \sqrt{u^2\sin^2\theta - 4\cdot\left(-\frac{1}{2}g\right)\cdot\left(\frac{v^2}{2g}\right)}}{2(-\frac{1}{2}g)} \\
 & = \frac{\pm u\sin\theta \mp \sqrt{u^2\sin^2\theta + v^2}}{g} \\
 & = \frac{\pm u\sin\theta + \sqrt{u^2\sin^2\theta + v^2}}{g}
\end{aligned}
$$

Since we always want the positive t.
Then the horizontal distance travelled will be

$$
\begin{aligned}
s &= u\cos\theta(t_1 + t_2) \\
&= u\cos\theta\frac{2\sqrt{u^2\sin^2\theta + v^2}}{g} \\
&= \frac{2u\cos\theta\sqrt{u^2\sin^2\theta + v^2}}{g} \\
s^2 &= \frac{4u^2}{g^2}\cos^2\theta(u^2\sin^2\theta + v^2) \\
&= \frac{4u^2}{g^2}\left(-u^2\cos^4\theta + (v^2+u^2)\cos^2\theta\right) \\
&= \frac{4u^2}{g^2}\left(-u^2\left(\cos^2\theta - \frac{v^2+u^2}{2u^2}\right)^2 + \frac{(v^2+u^2)^2}{4u^2}\right) \\
&= \frac{(v^2+u^2)^2}{g^2} - \frac{4u^4}{g^2}\left(\cos^2\theta - \frac{v^2+u^2}{2u^2}\right)^2
\end{aligned}
$$

If $u \geq v$ then such a θ exists such that we can achieve the maximum, ie $s = \frac{v^2+u^2}{g}$. If not, then we will achieve our maximum when $\cos\theta = 1$, ie $\sin\theta = 0$ and $s = \frac{2uv}{g}$

Question (1995 STEP I Q9)

A particle is projected from a point O with speed $\sqrt{2gh}$, where g is the acceleration due to gravity. Show that it is impossible, whatever the angle of projection, for the particle to reach a point above the parabola

$$x^2 = 4h(h-y),$$

where x is the horizontal distance from O and y is the vertical distance above O. State briefly the simplifying assumptions which this solution requires.

The position of the particle is projected at angle θ is $(x, y) = (v\cos\theta t, v\sin\theta t - \frac{1}{2}gt^2)$, ie $t = \frac{x}{v\cos\theta}$,

$$\begin{aligned} y &= x\tan\theta - \frac{1}{2}g\frac{x^2}{v^2}\sec^2\theta \\ y &= x\tan\theta - \frac{1}{4h}(1+\tan^2\theta)x^2 \\ 0 &= \frac{1}{4h}x^2\tan^2\theta - x\tan\theta + \frac{x^2}{4h} + y \\ \Delta \geq 0: \quad 0 &\leq \Delta = x^2 - 4\frac{x^2}{4h}\left(\frac{x^2}{4h} + y\right) \\ &= 1 - \frac{1}{4h^2}(x^2 + 4hy) \\ \Rightarrow \quad x^2 + 4hy &\leq 4h^2 \\ \Rightarrow \quad x^2 &\leq 4h(h-y) \end{aligned}$$

We are assuming that there are no forces acting other than gravity (eg air resistance)

Question (1995 STEP III Q10)

A cannon is situated at the bottom of a plane inclined at angle β to the horizontal. A (small) cannon ball is fired from the cannon at an initial speed u. Ignoring air resistance, find the angle of firing which will maximise the distance up the plane travelled by the cannon ball and show that in this case the ball will land at a distance

$$\frac{u^2}{g(1+\sin\beta)}$$

from the cannon.

Question (1996 STEP I Q11)

A particle is projected under the influence of gravity from a point O on a level plane in such a way that, when its horizontal distance from O is c, its height is h. It then lands on the plane at a distance $c+d$ from O. Show that the angle of projection α satisfies

$$\tan\alpha = \frac{h(c+d)}{cd}$$

and that the speed of projection v satisfies

$$v^2 = \frac{g}{2}\left(\frac{cd}{h} + \frac{(c+d)^2 h}{cd}\right).$$

Question (1997 STEP II Q11)

A tennis player serves from height H above horizontal ground, hitting the ball downwards with speed v at an angle α below the horizontal. The ball just clears the net of height h at horizontal distance a from the server and hits the ground a further horizontal distance b beyond the net. Show that

$$v^2 = \frac{g(a+b)^2(1+\tan^2\alpha)}{2[H-(a+b)\tan\alpha]}$$

and

$$\tan\alpha = \frac{2a+b}{a(a+b)}H - \frac{a+b}{ab}h\,.$$

By considering the signs of v^2 and $\tan\alpha$, find upper and lower bounds on H for such a serve to be possible.

Question (1998 STEP I Q10)

A shell explodes on the surface of horizontal ground. Earth is scattered in all directions with varying velocities. Show that particles of earth with initial speed v landing a distance r from the centre of explosion will do so at times t given by

$$\tfrac{1}{2}g^2t^2 = v^2 \pm \sqrt{(v^4 - g^2r^2)}.$$

Find an expression in terms of v, r and g for the greatest height reached by such particles.

Question (1998 STEP II Q11)

A fielder, who is perfectly placed to catch a ball struck by the batsman in a game of cricket, watches the ball in flight. Assuming that the ball is struck at the fielder's eye level and is caught just in front of her eye, show that $\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta)$ is constant, where θ is the angle between the horizontal and the fielder's line of sight. In order to catch the next ball, which is also struck towards her but at a different velocity, the fielder runs at constant speed v towards the batsman. Assuming that the ground is horizontal, show that the fielder should choose v so that $\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta)$ remains constant.

Set up a coordinate frame such that the position of the catch is the origin and the time of the catch is $t = 0$. We must have then that the trajectory of the ball is $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2 = \binom{u_x t}{u_y t - \frac{1}{2}gt^2}$.

We must then have:

$$\begin{aligned} \tan\theta &= \frac{u_y t - \frac{1}{2}gt^2}{u_x t} \\ &= \frac{u_y}{u_x} - \frac{g}{2u_x}t \\ \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta) &= 0 - \frac{g}{2u_x} \end{aligned}$$

which is clearly constant.

Set coordinates so y-axis starts from eye-level and $t = 0$ the first time the ball reaches that level. (Or move the trajectory backwards if that's not the case).

Then the ball has trajectory $\binom{u_x t}{u_y t - \frac{1}{2}gt^2}$. The ball reaches eye level a second time when $t = \frac{2u_y}{g}$, ie at a point $\frac{2u_x u_y}{g}$.

The fielder therefore needs to have position $f + (u_x - \frac{g}{2u_y}f)t$ at all times.

Therefore

$$\begin{aligned} \tan\theta &= \frac{u_y t - \frac{1}{2}gt^2}{f + (u_x - \frac{g}{2u_y}f)t - u_x t} \\ &= \frac{u_y t - \frac{1}{2}gt^2}{f(1 - \frac{g}{2u_y}t)} \\ &= \frac{u_y t(1 - \frac{g}{2u_y}t)}{f(1 - \frac{g}{2u_y}t)} \\ &= u_y t \\ \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta) &= u_y \end{aligned}$$

Ie $\frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta)$ is constant as required.

Question (2000 STEP I Q9)

A child is playing with a toy cannon on the floor of a long railway carriage. The carriage is moving horizontally in a northerly direction with acceleration a. The child points the cannon southward at an angle θ to the horizontal and fires a toy shell which leaves the cannon at speed V. Find, in terms of a and g, the value of $\tan 2\theta$ for which the cannon has maximum range (in the carriage). If a is small compared with g, show that the value of θ which gives the maximum range is approximately

$$\frac{\pi}{4} + \frac{a}{2g},$$

and show that the maximum range is approximately $\dfrac{V^2}{g} + \dfrac{V^2 a}{g^2}$.

$$s_x = V\cos\theta t + \frac{1}{2}at^2$$

$$
\begin{aligned}
s_y &= V\sin\theta t - \frac{1}{2}gt^2 \\
\Rightarrow \quad T &= \frac{2V\sin\theta}{g} \\
\Rightarrow \quad s_{max} &= \frac{2V^2\sin\theta\cos\theta}{g} + \frac{1}{2}a\frac{4V^2\sin^2\theta}{g^2} \\
&= (g\sin 2\theta + 2a\sin^2\theta)\frac{V^2}{g^2} \\
\frac{\mathrm{d}s_{max}}{\mathrm{d}\theta} &= (2g\cos 2\theta + 4a\cos\theta\sin\theta)\frac{V^2}{g^2} \\
&= (2g\cos 2\theta + 2a\sin 2\theta)\frac{V^2}{g^2} \\
\Rightarrow \quad \tan 2\theta &= -\frac{a}{g} \\
\Rightarrow \quad 2\theta &\in (\frac{\pi}{2}, \pi) \\
\Rightarrow \quad \tan\left(\frac{\pi}{2} - 2\theta\right) &= -\frac{a}{g} \\
\Rightarrow \quad \frac{\pi}{2} - 2\theta &\approx -\frac{a}{g} \\
\Rightarrow \quad \theta &\approx \frac{\pi}{4} + \frac{a}{2g}
\end{aligned}
$$

$$
\begin{aligned}
s_{max} &\approx \left(g\sin\left(\frac{\pi}{2} + \frac{a}{g}\right) + 2a\sin^2\left(\frac{\pi}{4} + \frac{a}{2g}\right)\right)\frac{V^2}{g^2} \\
&\approx \left(g\cdot 1 + 2a\left(\frac{1}{\sqrt{2}}(\frac{a}{2g} + 1)\right)^2\right)\frac{V^2}{g^2} \\
&\approx \left(g + a\left(1 + \frac{a}{g}\right)^2\right)\frac{V^2}{g^2} \\
&\approx (g + a)\frac{V^2}{g^2} \\
&= \frac{V^2}{g} + \frac{V^2 a}{g}
\end{aligned}
$$

Question (2001 STEP I Q10)

A gun is sited on a horizontal plain and can fire shells in any direction and at any elevation at speed v. The gun is a distance d from a straight railway line which crosses the plain, where $v^2 > gd$. The gunner aims to hit the line, choosing the direction and elevation so as to maximize the time of flight of the shell. Show that the time of flight, T, of the shell satisfies

$$g^2T^2 = 2v^2 + 2\left(v^4 - g^2d^2\right)^{\frac{1}{2}}.$$

Extension: (Not in original paper) Find the time of flight if the gun is constrained so that the angle of elevation α is not greater than 45°.

If we fire the gun at an angle to the track, as long as we can travel a horizontal distance

$\geq d$ we can hit the track. Suppose we am at an elevatation α, then

$$\begin{aligned}
(\uparrow): && s &= ut + \frac{1}{2}at^2 \\
&& 0 &= v\sin\alpha T - \frac{1}{2}gT^2 \\
\Rightarrow && T &= \frac{2v\sin\alpha}{g}
\end{aligned}$$

$$\begin{aligned}
(\rightarrow): && s &= ut \\
&& s &= v\cos\alpha T \\
&& &= v\sqrt{1-\sin^2\alpha}T \\
&& &= vT\sqrt{1-\frac{g^2T^2}{4v^2}} \\
&& &= \frac{T}{2}\sqrt{4v^2-g^2T^2} \\
\Rightarrow && d &\leq \frac{T}{2}\sqrt{4v^2-g^2T^2} \\
\Rightarrow && 4g^2d^2 &\leq g^2T^2(4v^2-g^2T^2) \\
\Rightarrow && 0 &\leq -(g^2T^2)^2 + 4v^2(g^2T^2) - 4g^2d^2 \\
&& &= 4v^4 - 4g^2d^2 - \left(g^2T^2-2v^2\right)^2 \\
\Rightarrow && \left(g^2T^2-2v^2\right)^2 &\leq 4v^4 - 4g^2d^2 \\
\Rightarrow && g^2T^2 &\leq 2v^2 + 2\sqrt{v^4-g^2d^2}
\end{aligned}$$

Therefore the maximum value for g^2T^2 is $2v^2 + 2\sqrt{v^4-g^2d^2}$

Notice that we are hitting the track directly at d. This is because to maximise the time of flight (for a fixed speed) we want to maximise the angle of elevation. Therefore we want the highest angle where we still hit the track (which is clearly the shortest distance).

If we are constraint to $\alpha \leq 45^\circ$ we know that T is maximised when $\alpha = 45^\circ$ (and we will reach the track since the range $\frac{v^2\sin 2\alpha}{g}$ is increasing). Therefore the maximum time is $T = \frac{\sqrt{2}v}{g}$

Question (2001 STEP II Q11)

A two-stage missile is projected from a point A on the ground with horizontal and vertical velocity components u and v, respectively. When it reaches the highest point of its trajectory an internal explosion causes it to break up into two fragments. Immediately after this explosion one of these fragments, P, begins to move vertically upwards with speed v_e, but retains the previous horizontal velocity. Show that P will hit the ground at a distance R from A given by

$$\frac{gR}{u} = v + v_e + \sqrt{v_e^2 + v^2}\,.$$

It is required that the range R should be greater than a certain distance D (where $D > 2uv/g$). Show that this requirement is satisfied if

$$v_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right).$$

[$\mathfrak{sl}\, Theeffectofairresistanceistobeneglected.$]

Question (2002 STEP II Q9)

A particle is projected from a point O on a horizontal plane with speed V and at an angle of elevation α. The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height h, at distances a and b from O. Given that the particle just passes over the walls, find $\tan\alpha$ in terms of a, b and h and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab}\,.$$

The heights of the walls are now increased by the same small positive amount δh. A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively. Show that

$$\sec^2\alpha\,\delta\alpha \approx \frac{a+b}{ab}\,\delta h\,,$$

and deduce that $\delta\alpha > 0$. Show also that δV is positive if $h > ab/(a+b)$ and negative if $h < ab/(a+b)$.

Question (2003 STEP I Q9)

A particle is projected with speed V at an angle θ above the horizontal. The particle passes through the point P which is a horizontal distance d and a vertical distance h from the point of projection. Show that

$$T^2 - 2kT + \frac{2kh}{d} + 1 = 0\,,$$

where $T = \tan\theta$ and $k = \dfrac{V^2}{gd}$. Show that, if $kd > h + \sqrt{h^2 + d^2}$, there are two distinct possible angles of projection.

Let these two angles be α and β. Show that $\alpha + \beta = \pi - \arctan(d/h)$.

Question (2003 STEP II Q11)

A particle P_1 is projected with speed V at an angle of elevation α $(> 45^\circ)$, from a point in a horizontal plane. Find T_1, the flight time of P_1, in terms of α, V and g. Show that the time after projection at which the direction of motion of P_1 first makes an angle of 45° with the horizontal is $\frac{1}{2}(1 - \cot\alpha)T_1$.

A particle P_2 is projected under the same conditions. When the direction of the motion of P_2 first makes an angle of 45° with the horizontal, the speed of P_2 is instantaneously doubled. If T_2 is the total flight time of P_2, show that

$$\frac{2T_2}{T_1} = 1 + \cot\alpha + \sqrt{1 + 3\cot^2\alpha}\,.$$

Question (2004 STEP I Q9)

A particle is projected over level ground with a speed u at an angle θ above the horizontal. Derive an expression for the greatest height of the particle in terms of u, θ and g.

A particle is projected from the floor of a horizontal tunnel of height $\frac{9}{10}d$. Point P is $\frac{1}{2}d$ metres vertically and d metres horizontally along the tunnel from the point of projection. The particle passes through point P and lands inside the tunnel without hitting the roof. Show that

$$\arctan\tfrac{3}{5} < \theta < \arctan 3\,.$$

Question (2005 STEP II Q10)

The points A and B are 180 metres apart and lie on horizontal ground. A missile is launched from A at speed of $100\,\mathrm{m\,s^{-1}}$ and at an acute angle of elevation to the line AB of $\arcsin\frac{3}{5}$. A time T seconds later, an anti-missile missile is launched from B, at speed of $200\,\mathrm{m\,s^{-1}}$ and at an acute angle of elevation to the line BA of $\arcsin\frac{4}{5}$. The motion of both missiles takes place in the vertical plane containing A and B, and the missiles collide. Taking $g = 10\,\mathrm{m\,s^{-2}}$ and ignoring air resistance, find T. [Note that $\arcsin\frac{3}{5}$ is another notation for $\sin^{-1}\frac{3}{5}$.]

Question (2006 STEP I Q10)

A particle P is projected in the x-y plane, where the y-axis is vertical and the x-axis is horizontal. The particle is projected with speed V from the origin at an angle of $45°$ above the positive x-axis. Determine the equation of the trajectory of P.

The point of projection (the origin) is on the floor of a barn. The roof of the barn is given by the equation $y = x\tan\alpha + b$, where $b > 0$ and α is an acute angle. Show that, if the particle just touches the roof, then $V(-1+\tan\alpha) = -2\sqrt{bg}$; you should justify the choice of the negative root. If this condition is satisfied, find, in terms of α, V and g, the time after projection at which touching takes place.

A particle Q can slide along a smooth rail fixed, in the x-y plane, to the under-side of the roof. It is projected from the point $(0, b)$ with speed U at the same time as P is projected from the origin. Given that the particles just touch in the course of their motions, show that

$$2\sqrt{2}\,U\cos\alpha = V\big(2 + \sin\alpha\cos\alpha - \sin^2\alpha\big).$$

Question (2006 STEP II Q11)

A projectile of unit mass is fired in a northerly direction from a point on a horizontal plain at speed u and an angle θ above the horizontal. It lands at a point A on the plain. In flight, the projectile experiences two forces: gravity, of magnitude g; and a horizontal force of constant magnitude f due to a wind blowing from North to South. Derive an expression, in terms of u, g, f and θ for the distance OA.

(i) Determine the angle α such that, for all $\theta > \alpha$, the wind starts to blow the projectile back towards O before it lands at A.

(ii) An identical projectile, which experiences the same forces, is fired from O in a northerly direction at speed u and angle $45°$ above the horizontal and lands at a point B on the plain. Given that θ is chosen to maximise OA, show that

$$\frac{OB}{OA} = \frac{g-f}{\sqrt{g^2+f^2}-f}\,.$$

Describe carefully the motion of the second projectile when $f = g$.

Question (2007 STEP I Q11)

A smooth, straight, narrow tube of length L is fixed at an angle of $30°$ to the horizontal. A particle is fired up the tube, from the lower end, with initial velocity u. When the particle reaches the upper end of the tube, it continues its motion until it returns to the same level as the lower end of the tube, having travelled a horizontal distance D after leaving the tube. Show that D satisfies the equation

$$4gD^2 - 2\sqrt{3}\left(u^2 - Lg\right)D - 3L\left(u^2 - gL\right) = 0$$

and hence that

$$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\,(u^2 - gL)}.$$

The final horizontal displacement of the particle from the lower end of the tube is R. Show that $\dfrac{\mathrm{d}R}{\mathrm{d}L} = 0$ when $2D = L\sqrt{3}$, and determine, in terms of u and g, the corresponding value of R.

Question (2007 STEP II Q9)

A solid right circular cone, of mass M, has semi-vertical angle α and smooth surfaces. It stands with its base on a smooth horizontal table. A particle of mass m is projected so that it strikes the curved surface of the cone at speed u. The coefficient of restitution between the particle and the cone is e. The impact has no rotational effect on the cone and the cone has no vertical velocity after the impact.

(i) The particle strikes the cone in the direction of the normal at the point of impact. Explain why the trajectory of the particle immediately after the impact is parallel to the normal to the surface of the cone. Find an expression, in terms of M, m, α, e and u, for the speed at which the cone slides along the table immediately after impact.

(ii) If instead the particle falls vertically onto the cone, show that the speed w at which the cone slides along the table immediately after impact is given by

$$w = \frac{mu(1+e)\sin\alpha\cos\alpha}{M + m\cos^2\alpha}\,.$$

Show also that the value of α for which w is greatest is given by

$$\cos\alpha = \sqrt{\frac{M}{2M+m}}\,.$$

Question (2007 STEP II Q11)

sl $Inthisquestiontaketheaccelerationduetogravitytobe 10\,\mathrm{m\ s^{-2}}$ and neglect air resistance. The point O lies in a horizontal field. The point B lies 50 m east of O. A particle is projected from B at speed $25\,\mathrm{m}\,s^{-1}$ at an angle $\arctan\frac{1}{2}$ above the horizontal and in a direction that makes an angle 60° with OB; it passes to the north of O.

Taking unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the directions east, north and vertically upwards, respectively, find the position vector of the particle relative to O at time t seconds after the particle was projected, and show that its distance from O is

$$5(t^2 - \sqrt{5}t + 10)\,\mathrm{m}.$$

When this distance is shortest, the particle is at point P. Find the position vector of P and its horizontal bearing from O.

(ii) Show that the particle reaches its maximum height at P.

(iii) When the particle is at P, a marksman fires a bullet from O directly at P. The initial speed of the bullet is $350\,\mathrm{m}\,s^{-1}$. Ignoring the effect of gravity on the bullet show that, when it passes through P, the distance between P and the particle is approximately 3 m.

Question (2007 STEP III Q10)

A particle is projected from a point on a plane that is inclined at an angle ϕ to the horizontal. The position of the particle at time t after it is projected is (x, y), where $(0, 0)$ is the point of projection, x measures distance up the line of greatest slope and y measures perpendicular distance from the plane. Initially, the velocity of the particle is given by $(\dot{x}, \dot{y}) = (V\cos\theta, V\sin\theta)$, where $V > 0$ and $\phi + \theta < \pi/2$. Write down expressions for x and y.

The particle bounces on the plane and returns along the same path to the point of projection. Show that

$$2\tan\phi\tan\theta = 1$$

and that

$$R = \frac{V^2\cos^2\theta}{2g\sin\phi},$$

where R is the range along the plane. Show further that

$$\frac{2V^2}{gR} = 3\sin\phi + \mathrm{cosec}\,\phi$$

and deduce that the largest possible value of R is $V^2/(\sqrt{3}\,g)$.

Question (2008 STEP II Q9)

In this question, use $g = 10\,\text{m}\,\text{s}^{-2}$. In cricket, a fast bowler projects a ball at $40\,\text{m}\,\text{s}^{-1}$ from a point $h\,\text{m}$ above the ground, which is horizontal, and at an angle α above the horizontal. The trajectory is such that the ball will strike the stumps at ground level a horizontal distance of 20 m from the point of projection.

(i) Determine, in terms of h, the two possible values of $\tan\alpha$. Explain which of these two values is the more appropriate one, and deduce that the ball hits the stumps after approximately half a second.

(ii) State the range of values of h for which the bowler projects the ball below the horizontal.

(iii) In the case $h = 2.5$, give an approximate value in degrees, correct to two significant figures, for α. You need not justify the accuracy of your approximation.

[You may use the small-angle approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.]

Question (2009 STEP I Q9)

Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection *above* the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection *below* the horizontal is β. The particles collide at time T after projection. Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation

$$(k^2 - 1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0\,.$$

Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that

$$gd \le (1+k)V^2\,.$$

Question (2009 STEP III Q9)

A particle is projected under gravity from a point P and passes through a point Q. The angles of the trajectory with the positive horizontal direction at P and at Q are θ and ϕ, respectively. The angle of elevation of Q from P is α.

(i) Show that $\tan\theta + \tan\phi = 2\tan\alpha$.

(ii) It is given that there is a second trajectory from P to Q with the same speed of projection. The angles of this trajectory with the positive horizontal direction at P and at Q are θ' and ϕ', respectively. By considering a quadratic equation satisfied by $\tan\theta$, show that $\tan(\theta+\theta') = -\cot\alpha$. Show also that $\theta + \theta' = \pi + \phi + \phi'$.

Question (2010 STEP II Q9)

Two points A and B lie on horizontal ground. A particle P_1 is projected from A towards B at an acute angle of elevation α and simultaneously a particle P_2 is projected from B towards A at an acute angle of elevation β. Given that the two particles collide in the air a horizontal distance b from B, and that the collision occurs after P_1 has attained its maximum height h, show that

$$2h\cot\beta < b < 4h\cot\beta$$

and

$$2h\cot\alpha < a < 4h\cot\alpha\,,$$

where a is the horizontal distance from A to the point of collision.

Question (2011 STEP I Q9)

A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_{d+}d_2^2}{d_d\,.}$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

Question (2011 STEP II Q10)

A particle is projected from a point on a horizontal plane, at speed u and at an angle θ above the horizontal. Let H be the maximum height of the particle above the plane. Derive an expression for H in terms of u, g and θ. A particle P is projected from a point O on a smooth horizontal plane, at speed u and at an angle θ above the horizontal. At the same instant, a second particle R is projected horizontally from O in such a way that R is vertically below P in the ensuing motion. A light inextensible string of length $\frac{1}{2}H$ connects P and R. Show that the time that elapses before the string becomes taut is

$$(\sqrt{2}-1)\sqrt{H/g}\,.$$

When the string becomes taut, R leaves the plane, the string remaining taut. Given that P and R have equal masses, determine the total horizontal distance, D, travelled by R from the moment its motion begins to the moment it lands on the plane again, giving your answer in terms of u, g and θ. Given that $D = H$, find the value of $\tan\theta$.

Question (2012 STEP I Q9)

A tall shot-putter projects a small shot from a point 2.5 m above the ground, which is horizontal. The speed of projection is $10\,\text{ms}^{-1}$ and the angle of projection is θ above the horizontal. Taking the acceleration due to gravity to be $10\,\text{ms}^{-2}$, show that the time, in seconds, that elapses before the shot hits the ground is

$$\frac{1}{\sqrt{2}}\left(\sqrt{1-c}+\sqrt{2-c}\right),$$

where $c = \cos 2\theta$. Find an expression for the range in terms of c and show that it is greatest when $c = \frac{1}{5}$. Show that the extra distance attained by projecting the shot at this angle rather than at an angle of 45° is $5(\sqrt{6}-\sqrt{2}-1)$ m.

$$\begin{aligned}
 && s &= ut + \frac{1}{2}gt^2 \\
\Rightarrow && -2.5 &= 10\sin\theta\, T - 5T^2 \\
\Rightarrow && T &= \frac{10\sin\theta \pm \sqrt{100\sin^2\theta - 4\cdot 5\cdot(-2.5)}}{10} \\
 && &= \sin\theta + \sqrt{\sin^2\theta + \frac{1}{2}} \\
 && &= \frac{1}{\sqrt{2}}\left(\sqrt{2}\sin\theta + \sqrt{2\sin^2\theta + 1}\right) \\
 && &= \frac{1}{\sqrt{2}}\left(\sqrt{2(1-\cos^2\theta)} + \sqrt{2-\cos 2\theta}\right) \\
 && &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\cos 2\theta} + \sqrt{2-\cos 2\theta}\right) \\
 && &= \frac{1}{\sqrt{2}}\left(\sqrt{1-c} + \sqrt{2-c}\right)
\end{aligned}$$

$$\begin{aligned}
s &= 10\cos\theta T \\
&= 10\sqrt{\frac{\cos 2\theta + 1}{2}}\,\frac{1}{\sqrt{2}}\left(\sqrt{1-c}+\sqrt{2-c}\right) \\
&= 5\sqrt{c+1}\left(\sqrt{1-c}+\sqrt{2-c}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{5}\frac{\mathrm{d}s}{\mathrm{d}c} &= \frac{1}{2}(c+1)^{-1/2}((1-c)^{1/2}+(2-c)^{1/2}) - \frac{1}{2}(c+1)^{1/2}\Big((1-c)^{-1/2}+(2-c)^{-1/} \\
&= \frac{((1-c)(2-c)^{1/2}+(2-c)(1-c)^{1/2}) - ((c+1)(2-c)^{1/2}+(c+1)(1-c)^{1/}}{2\sqrt{c+1}\sqrt{1-c}\sqrt{2-c}} \\
&= \frac{\sqrt{2-c}\,(1-c-c-1)+\sqrt{1-c}\,(2-c-c-1))}{2\sqrt{c+1}\sqrt{1-c}\sqrt{2-c}} \\
&= \frac{\sqrt{1-c}\,(1-2c)-2c\sqrt{2-c}}{2\sqrt{c+1}\sqrt{1-c}\sqrt{2-c}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}c} = 0: && \sqrt{1-c}\,(1-2c) &= 2c\sqrt{2-c} \\
\Rightarrow && (1-c)(1-2c)^2 &= 4c^2(2-c)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad 1 - 5c + 8c^2 - 4c^3 &= 8c^2 - 4c^3 \\
\Rightarrow \quad 0 &= -5c + 1 \\
\Rightarrow \quad c &= \frac{1}{5}
\end{aligned}$$

When $\theta = 45°, c = 0$, so $s_{45°} = 5(1+\sqrt{2})$
When $c = \frac{1}{5}$,

$$\begin{aligned}
s &= 5\sqrt{\frac{1}{5}+1}\left(\sqrt{1-\frac{1}{5}} + \sqrt{2-\frac{1}{5}}\right) \\
&= 5\sqrt{\frac{6}{5}}\left(\sqrt{\frac{4}{5}} + \sqrt{\frac{9}{5}}\right) \\
&= 2\sqrt{6} + 3\sqrt{6} = 5\sqrt{6}
\end{aligned}$$

Therefore the additional distance is $5(\sqrt{6} - \sqrt{2} - 1)$

Question (2012 STEP II Q9)

A tennis ball is projected from a height of $2h$ above horizontal ground with speed u and at an angle of α below the horizontal. It travels in a plane perpendicular to a vertical net of height h which is a horizontal distance of a from the point of projection. Given that the ball passes over the net, show that

$$\frac{1}{u^2} < \frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}\,.$$

The ball lands before it has travelled a horizontal distance of b from the point of projection. Show that

$$\sqrt{u^2\sin^2\alpha + 4gh} < \frac{bg}{u\cos\alpha} + u\sin\alpha\,.$$

Hence show that

$$\tan\alpha < \frac{h(b^2 - 2a^2)}{ab(b-a)}\,.$$

$$\begin{aligned}
s &= ut \\
\Rightarrow \quad a &= u\cos\alpha t \\
\Rightarrow \quad t &= \frac{a}{u\cos\alpha} \\
s &= ut + \frac{1}{2}at^2 \\
\Rightarrow \quad -h &< -u\sin\alpha\frac{a}{u\cos\alpha} - \frac{1}{2}g\left(\frac{a}{u\cos\alpha}\right)^2 \\
&= -a\tan\alpha - \frac{1}{2}ga^2\frac{1}{u^2}\sec^2\alpha \\
\Rightarrow \quad \frac{1}{2}ga^2\frac{1}{u^2}\sec^2\alpha &< h - a\tan\alpha \\
\Rightarrow \quad \frac{1}{u^2} &< \frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}
\end{aligned}$$

$$
\begin{aligned}
& & s &= ut + \frac{1}{2}at^2 \\
&\Rightarrow & 2h &= u\sin\alpha t + \frac{1}{2}gt^2 \\
&\Rightarrow & t &= \frac{-u\sin\alpha \pm \sqrt{u^2\sin^2\alpha + 4hg}}{g} \\
& & t &= \frac{-u\sin\alpha + \sqrt{u^2\sin^2\alpha + 4hg}}{g} \\
& & s &= ut \\
&\Rightarrow & b &> u\cos\alpha t \\
&\Rightarrow & \frac{b}{u\cos\alpha} &> \frac{-u\sin\alpha + \sqrt{u^2\sin^2\alpha + 4hg}}{g} \\
&\Rightarrow & \sqrt{u^2\sin^2\alpha + 4hg} &< \frac{bg}{u\cos\alpha} + u\sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow & u^2\sin^2\alpha + 4hg &< \frac{b^2g^2}{u^2\cos^2\alpha} + u^2\sin^2\alpha + 2bg\tan\alpha \\
&\Rightarrow & 4hg - 2bg\tan\alpha &< \frac{b^2g^2}{u^2\cos^2\alpha} \\
& & &< \frac{b^2g^2}{\cos^2\alpha}\frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha} \\
& & &= \frac{2b^2g(h - a\tan\alpha)}{a^2} \\
&\Rightarrow & \tan\alpha\left(\frac{2b^2g}{a} - 2bg\right) &< \frac{2b^2gh}{a^2} - 4hg \\
&\Leftrightarrow & \tan\alpha\left(\frac{2b^2g - 2abg}{a}\right) &< \frac{2b^2gh - 4hga^2}{a^2} \\
&\Leftrightarrow & \tan\alpha\left(\frac{2bg(b-a)}{a}\right) &< \frac{2hg(b^2 - 2a^2)}{a^2} \\
&\Rightarrow & \tan\alpha &< \frac{h(b^2 - 2a^2)}{ab(b-a)}
\end{aligned}
$$

Question (2013 STEP I Q9)

Two particles, A and B, are projected simultaneously towards each other from two points which are a distance d apart in a horizontal plane. Particle A has mass m and is projected at speed u at angle α above the horizontal. Particle B has mass M and is projected at speed v at angle β above the horizontal. The trajectories of the two particles lie in the same vertical plane. The particles collide directly when each is at its point of greatest height above the plane. Given that both A and B return to their starting points, and that momentum is conserved in the collision, show that

$$m \cot \alpha = M \cot \beta \,.$$

Show further that the collision occurs at a point which is a horizontal distance b from the point of projection of A where

$$b = \frac{Md}{m+M}\,,$$

and find, in terms of b and α, the height above the horizontal plane at which the collision occurs.

Since A and B return to their starting points, and at their highest points there is no vertical component to their velocities, their horizontal must perfectly reverse, ie

$$\begin{aligned} mu\cos\alpha - Mv\cos\beta &= -mu\cos\alpha + Mv\cos\beta \\ \Rightarrow \qquad mu\cos\alpha &= Mv\cos\beta \end{aligned}$$

Since they reach their highest points at the same time, they must have the same initial vertical speed, ie $u\sin\alpha = v\sin\beta$, so

$$\begin{aligned} mv\frac{\sin\beta}{\sin\alpha}\cos\alpha &= Mv\cos\beta \\ \Rightarrow \qquad m\cot\alpha &= M\cot\beta \end{aligned}$$

The horizontal distance travelled by A B will be:

$$\begin{aligned} d_A &= u\cos\alpha t \\ d_B &= v\cos\beta t \\ \Rightarrow \qquad \frac{d_A}{d_A+d_B} &= \frac{u\cos\alpha}{u\cos\alpha + v\cos\beta} \\ &= \frac{\frac{M}{m}v\cos\beta}{\frac{M}{m}v\cos\beta + v\cos\beta} \\ &= \frac{M}{M+m} \\ \Rightarrow \qquad d_A = b &= \frac{Md}{m+M} \end{aligned}$$

Applying $v^2 = u^2 + 2as$ we see that

$$0 = u\sin\alpha - gt$$

$$
\begin{aligned}
\Rightarrow \qquad t &= \frac{u\sin\alpha}{g} \\
b &= u\cos\alpha\frac{u\sin\alpha}{g} \\
\Rightarrow \qquad u^2 &= \frac{2bg}{\sin 2\alpha} \\
0 &= u^2\sin^2\alpha - 2gh \\
\Rightarrow \qquad h &= \frac{u^2\sin^2\alpha}{2g} \\
&= \frac{2bg}{\sin 2\alpha}\frac{\sin^2\alpha}{2g} \\
&= \frac{1}{2}b\tan\alpha
\end{aligned}
$$

Question (2013 STEP II Q10)

A particle is projected at an angle of elevation α (where $\alpha > 0$) from a point A on horizontal ground. At a general point in its trajectory the angle of elevation of the particle from A is θ and its direction of motion is at an angle ϕ above the horizontal (with $\phi \geq 0$ for the first half of the trajectory and $\phi \leq 0$ for the second half). Let B denote the point on the trajectory at which $\theta = \frac{1}{2}\alpha$ and let C denote the point on the trajectory at which $\phi = -\frac{1}{2}\alpha$.

(i) Show that, at a general point on the trajectory, $2\tan\theta = \tan\alpha + \tan\phi$.

(ii) Show that, if B and C are the same point, then $\alpha = 60°$.

(iii) Given that $\alpha < 60°$, determine whether the particle reaches the point B first or the point C first.

Question (2014 STEP I Q9)

A particle of mass m is projected due east at speed U from a point on horizontal ground at an angle θ above the horizontal, where $0 < \theta < 90°$. In addition to the gravitational force mg, it experiences a horizontal force of magnitude mkg, where k is a positive constant, acting due west in the plane of motion of the particle. Determine expressions in terms of U, θ and g for the time, T_H, at which the particle reaches its greatest height and the time, T_L, at which it lands. Let $T = U\cos\theta/(kg)$. By considering the relative magnitudes of T_H, T_L and T, or otherwise, sketch the trajectory of the particle in the cases $k\tan\theta < \frac{1}{2}$, $\frac{1}{2} < k\tan\theta < 1$, and $k\tan\theta > 1$. What happens when $k\tan\theta = 1$?

Question (2014 STEP II Q10)

A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x-y plane, where the x-axis is horizontal, the y-axis is vertical and the origin is O. Obtain the Cartesian equation of the particle's trajectory in terms of u, g and λ, where $\lambda = \tan\theta$.

Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x, the coordinate y can take all values up to a maximum value, Y, which you should determine as a function of x, u and g. Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u. Hence find the furthest distance from O that can be achieved by such a projectile.

Question (2015 STEP I Q9)

A short-barrelled machine gun stands on horizontal ground. The gun fires bullets, from ground level, at speed u continuously from $t = 0$ to $t = \dfrac{\pi}{6\lambda}$, where λ is a positive constant, but does not fire outside this time period. During this time period, the angle of elevation α of the barrel decreases from $\frac{1}{3}\pi$ to $\frac{1}{6}\pi$ and is given at time t by

$$\alpha = \tfrac{1}{3}\pi - \lambda t\,.$$

Let $k = \dfrac{g}{2\lambda u}$. Show that, in the case $\frac{1}{2} \leq k \leq \frac{1}{2}\sqrt{3}$, the last bullet to hit the ground does so

at a distance

$$\frac{2ku^2\sqrt{1-k^2}}{g}$$

from the gun.

What is the corresponding result if $k < \frac{1}{2}$?

Question (2016 STEP I Q11)

The point O is at the top of a vertical tower of height h which stands in the middle of a large horizontal plain. A projectile P is fired from O at a fixed speed u and at an angle α above the horizontal. Show that the distance x from the base of the tower when P hits the plain satisfies

$$\frac{gx^2}{u^2} = h(1+\cos 2\alpha) + x\sin 2\alpha\,.$$

Show that the greatest value of x as α varies occurs when $x = h\tan 2\alpha$ and find the corresponding value of $\cos 2\alpha$ in terms of g, h and u. Show further that the greatest achievable distance between O and the landing point is $\dfrac{u^2}{g} + h\,.$

$\rightarrow$: $$x = u\cos\alpha t$$

$$\Rightarrow \qquad t = \frac{x}{u\cos\alpha}$$

$$\uparrow: \qquad -h = u\sin\alpha t - \frac{1}{2}gt^2$$

$$-h = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2}\sec^2\alpha$$

$$\Rightarrow \qquad \frac{gx^2}{u^2} = h(2\cos^2\alpha) + x2\tan\alpha\cos^2\alpha$$

$$= h(1+\cos 2\alpha) + x\sin 2\alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}: \qquad \frac{g}{u^2}2x\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -2h\sin 2\alpha + 2x\cos 2\alpha + \frac{\mathrm{d}x}{\mathrm{d}\alpha}\sin 2\alpha$$

$$\Rightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}\alpha}\left(\frac{2xg}{u^2} - \sin 2\alpha\right) = 2\cos 2\alpha(x - h\tan 2\alpha)$$

Since the turning point will be a maximum must be $x = h\tan 2\alpha$. Therefore, let $c = \cos 2\alpha$

$$\frac{gh^2}{u^2}\tan^2 2\alpha = h(1+\cos 2\alpha) + h\tan 2\alpha\sin 2\alpha$$

$$\Rightarrow \qquad \frac{gh}{u^2}(c^{-2}-1) = 1 + c + \frac{1-c^2}{c}$$

$$\Rightarrow \qquad \frac{gh(1-c^2)}{u^2c^2} = \frac{c+c^2+1-c^2}{c}$$

$$= \frac{1+c}{c}$$

$$\Rightarrow \qquad \frac{gh(1-c)}{u^2c} = 1$$

$$\Rightarrow \qquad u^2c = gh(1-c)$$

$$\Rightarrow \qquad c(u^2+gh) = gh$$

$$\Rightarrow \qquad \cos 2\alpha = \frac{gh}{u^2+gh}$$

$$\Rightarrow \qquad d_{max}^2 = h^2 + h^2\tan^2 2\alpha$$

$$= h^2\sec^2 2\alpha$$

$$= h^2\frac{(u^2+gh)^2}{g^2h^2}$$

$$= \frac{(u^2+gh)^2}{g^2}$$

$$= \left(\frac{u^2}{g} + h\right)^2$$

$$\Rightarrow \qquad d_{max} = \frac{u^2}{g} + h$$

Question (2016 STEP II Q11) **(i)** Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are $(a + ut\cos\alpha\,,\, ut\sin\alpha)$ and $(vt\cos\beta\,,\, b + vt\sin\beta)$, where a, b, u and v are positive constants, α and β are constant acute angles, and $t \geq 0$.

Given that the two particles collide, show that

$$u\sin(\theta + \alpha) = v\sin(\theta + \beta)\,,$$

where θ is the acute angle satisfying $\tan\theta = \dfrac{b}{a}$.

(ii) A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β. Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α, in a direction directly away from the tower.

Given that the target is hit before it reaches the ground, show that

$$2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg\,.$$

Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Question (2017 STEP I Q9)

A particle is projected at speed u from a point O on a horizontal plane. It passes through a fixed point P which is at a horizontal distance d from O and at a height $d\tan\beta$ above the plane, where $d > 0$ and β is an acute angle. The angle of projection α is chosen so that u is as small as possible.

(i) Show that $u^2 = gd\tan\alpha$ and $2\alpha = \beta + 90°$.

(ii) At what angle to the horizontal is the particle travelling when it passes through P? Express your answer in terms of α in its simplest form.

(i)

$$\begin{aligned}
d &= u\cos\alpha t \\
d\tan\beta &= u\sin\alpha t - \frac{1}{2}gt^2 \\
&= d\tan\alpha - \frac{1}{2u^2}gd^2\sec^2\alpha \\
\Rightarrow \qquad u^2 &= \frac{gd\sec^2\alpha}{2(\tan\alpha + \tan\beta)} \\
&= \frac{gdt^2}{2(t + \tan\beta)} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(u^2\right) &= \frac{2gdt\cdot 2(t+\tan\beta) - gdt^2\cdot 2}{4(t+\tan\beta)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2gdt(2t - t + 2\tan\beta)}{4(t + \tan\beta)^2} \\
&= \frac{gdt(t + 2\tan\beta)}{2(t + \tan\beta)^2}
\end{aligned}
$$

So either $t = 0$ or $t = -2\tan\beta$

$$
\begin{aligned}
u^2 &= \frac{gd \cdot 4\tan^2\beta}{2(-2\tan\beta + \tan\beta)} \\
&= \frac{2gd\tan\beta}{-1} \\
&= gd(-2\tan\beta) \\
&= gd\tan\alpha
\end{aligned}
$$

$$
\begin{aligned}
d\tan\beta &= d\tan\alpha - \frac{1}{2}\frac{gd^2}{gd\tan\alpha \cdot \cos^2\alpha} \\
\Rightarrow \quad \tan\beta &= \tan\alpha - \frac{1}{2\sin\alpha\cos\alpha} \\
&= \frac{2\sin^2\alpha - 1}{2\sin\alpha\cos\alpha} \\
&= \frac{-\cos 2\alpha}{\sin 2\alpha} \\
&= -\cot 2\alpha \\
&= \tan(2\alpha - 90^\circ) \\
\Rightarrow \quad \beta &= 2\alpha - 90^\circ \\
\Rightarrow \quad 2\alpha &= \beta + 90^\circ
\end{aligned}
$$

(ii) Suppose the angle to the horizontal is θ, then $\tan\theta = \frac{v_y}{v_x}$ so

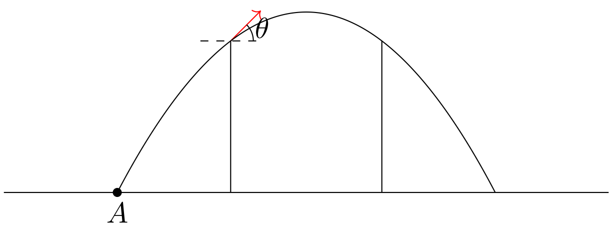
$$
\begin{aligned}
\tan\theta &= \frac{u\sin\alpha - gt}{u\cos\alpha} \\
&= \frac{u\sin\alpha - \frac{1}{2}g\frac{d}{u\cos\alpha}}{u\cos\alpha} \\
&= \frac{u^2\sin\alpha\cos\alpha - gd}{u^2\cos^2\alpha} \\
&= \frac{gd\tan\alpha\sin\alpha\cos\alpha - gd}{gd\tan\alpha \cdot \cos^2\alpha} \\
&= \frac{\tan\alpha\sin\alpha\cos\alpha - 1}{\sin\alpha\cos\alpha} \\
&= \frac{\sin^2\alpha - 1}{\sin\alpha\cos\alpha} \\
&= -\frac{\cos\alpha}{\sin\alpha} \\
&= -\cot\alpha = \tan(\alpha - 90^\circ) \\
\Rightarrow \quad \theta &= \alpha - 90^\circ
\end{aligned}
$$

Question (2017 STEP II Q11)

Two thin vertical parallel walls, each of height $2a$, stand a distance a apart on horizontal ground. The projectiles in this question move in a plane perpendicular to the walls.

(i) A particle is projected with speed $\sqrt{5ag}$ towards the two walls from a point A at ground level. It just clears the first wall. By considering the energy of the particle, find its speed when it passes over the first wall. Given that it just clears the second wall, show that the angle its trajectory makes with the horizontal when it passes over the first wall is $45°$. Find the distance of A from the foot of the first wall.

(ii) A second particle is projected with speed $\sqrt{5ag}$ from a point B at ground level towards the two walls. It passes a distance h above the first wall, where $h > 0$. Show that it does not clear the second wall.



(i)

$$\textbf{COE}: \quad \frac{1}{2}m \cdot 5ag = mg \cdot 2a + \frac{1}{2}mv^2$$
$$\Rightarrow \quad v^2 = ag$$
$$v = \sqrt{ag}$$

If it just clears the second wall, we must have:

$$0 = \sqrt{ag}\sin\theta t - \frac{1}{2}gt^2$$
$$\Rightarrow \quad t = \frac{2\sqrt{ag}\sin\theta}{g}$$
$$a = \sqrt{ag}\cos\theta t$$
$$= \sqrt{ag}\cos\theta\frac{2\sqrt{ag}\sin\theta}{g}$$
$$= a\sin 2\theta$$
$$\Rightarrow \quad \theta = 45°$$

Imagine firing the particle backwards from the top of the wall at $45°$ then

$$-2a = \sqrt{ag} \cdot \left(-\frac{1}{\sqrt{2}}\right) t - \frac{1}{2}gt^2$$

$$\begin{aligned}
\Rightarrow \quad 0 &= gt^2 + \sqrt{2ag}t - 4a \\
&= (\sqrt{g}t - \sqrt{2}\sqrt{a})(\sqrt{g}t + 2\sqrt{2}\sqrt{a}) \\
\Rightarrow \quad t &= \sqrt{\frac{2a}{g}} \\
\Rightarrow \quad s &= \left(-\frac{1}{\sqrt{2}}\right)\sqrt{ag}\sqrt{\frac{2a}{g}} \\
&= -a
\end{aligned}$$

Therefore the A is a from the wall.

(ii) When it passes over the first wall,

$$\begin{aligned}
\textbf{COE}: \quad \frac{5}{2}amg &= (2a+h)mg + \frac{1}{2}mv^2 \\
\Rightarrow \quad v^2 &= (a-2h)g
\end{aligned}$$

Now imagine firing a particle with this speed in any direction. The question is asking whether we can ever travel $2a$ without descending more than h.

$$\begin{aligned}
a &= \sqrt{(a-2h)g}\cos\beta t \\
\Rightarrow \quad t &= \frac{a}{\sqrt{(a-2h)g}\cos\beta} \\
-h &= \sqrt{(a-2h)g}\sin\beta t - \frac{1}{2}gt^2 \\
&= a\tan\beta - \frac{1}{2}\frac{a^2}{(a-2h)}\sec^2\beta \\
&= a\tan\beta - \frac{a^2}{2(a-2h)}(1+\tan^2\beta) \\
\Rightarrow \quad 0 &= \frac{a^2}{2(a-2h)}\tan^2\beta - a\tan\beta + \frac{a^2-2ah+4h^2}{2(a-2h)} \\
\Delta &= a^2 - \frac{a^2}{a-2h}\frac{a^2-2ah+4h^2}{a-2h} \\
&= \frac{a^2}{(a-2h)^2}\left(a^2-4ah+4h^2-a^2+2ah-4h^2\right) \\
&= \frac{a^2}{(a-2h)^2}(-2ah) < 0
\end{aligned}$$

So there are no solutions if $h > 0$

Question (2019 STEP I Q10)

In this question, the x-axis is horizontal and the positive y-axis is vertically upwards. A particle is projected from the origin with speed u at an angle α *to the vertical.* The particle passes through the fixed point $(h \tan \beta, h)$, where $0 < \beta < 90^\circ$ and $h > 0$.

(i) Show that

$$c^2 - ck \cot \beta + 1 + k \cot^2 \beta = 0, \quad (*)$$

where $c = \cot \alpha$ and $k = \frac{2u^2}{gh}$. You are given that there are two distinct values of α that satisfy equation $(*)$. Let α_1 and α_2 be these values.

a) Show that

$$\cot \alpha_1 + \cot \alpha_2 = k \cot \beta.$$

Show also that

$$\alpha_1 + \alpha_2 = \beta.$$

b) Show that

$$k > 2(1 + \sec \beta).$$

(ii) By considering the greatest height attained by the particle, show that $k \geq 4 \sec^2 \alpha$.

(i) The horizontal position of the particle at time t is $u \sin \alpha t$, so $T = \frac{h \tan \beta}{u \sin \alpha}$

The vertical position of the particle at this time T satisifes:

$$\begin{aligned} h &= u \cos \alpha \frac{h \tan \beta}{u \sin \alpha} - \frac{1}{2} g \left(\frac{h \tan \beta}{u \sin \alpha} \right)^2 \\ &= h \cot \alpha \tan \beta - \frac{gh^2}{2u^2} \tan^2 \beta^2 \beta \\ \Rightarrow \quad 1 &= c \tan \beta - \frac{1}{k} \tan^2 \beta (1 + c^2) \\ \Rightarrow \quad k \cot^2 \beta &= kc \cot \beta - 1 - c^2 \\ \Rightarrow \quad 0 &= c^2 - ck \cot \beta + 1 + k \cot^2 \beta \end{aligned}$$

a) As a quadratic in c the sum of the roots is $k \cot \beta$, therefore $\cot \alpha_1 + \cot \alpha_2 = k \cot \beta$. We also have that $\cot \alpha_1 \cot \alpha_2 = 1 + k \cot^2 \beta$, so

$$\begin{aligned} \cot(\alpha_1 + \alpha_2) &= \frac{\cot \alpha_1 \cot \alpha_2 - 1}{\cot \alpha_1 + \cot \alpha_2} \\ &= \frac{1 + k \cot^2 \beta - 1}{k \cot \beta} \\ &= \cot \beta \\ \Rightarrow \quad \beta &= \alpha_1 + \alpha_2 \pmod{\pi} \end{aligned}$$

but since $\alpha_i \in (0, \frac{\pi}{2})$ the equation must hold exactly.

b) Since it has two real roots we must have

$$\begin{aligned} 0 < \Delta &= k^2 \cot^2 \beta - 4(1 + k\cot^2 \beta) \\ &= k^2 \cot^2 \beta - 4k \cot^2 \beta - 4 \\ &= \cot^2 \beta(k^2 - 4k - 4(\sec^2 \beta - 1)) \\ &= \cot^2 \beta((k-2)^2 - 4\sec^2 \beta) \\ \Rightarrow \quad k &> 2 + 2\sec \beta = 2(1 + \sec \beta) \end{aligned}$$

(ii) The greatest height will satisfy $v^2 = u^2 + 2as$ so $0 = u^2\cos^2\alpha - 2gh_{max} \Rightarrow 4\sec^2\alpha = \frac{2u^2}{gh_{max}} = k_{max}$, but this decreases with h, so the smallest k can be is $4\sec^2\alpha$, ie $k \geq 4\sec^2\alpha$

Question (2019 STEP II Q9)

A particle P is projected from a point O on horizontal ground with speed u and angle of projection α, where $0 < \alpha < \frac{1}{2}\pi$.

(i) Show that if $\sin\alpha < \frac{2\sqrt{2}}{3}$, then the distance OP is increasing throughout the flight. Show also that if $\sin\alpha > \frac{2\sqrt{2}}{3}$, then OP will be decreasing at some time before the particle lands.

(ii) At the same time as P is projected, a particle Q is projected horizontally from O with speed v along the ground in the opposite direction from the trajectory of P. The ground is smooth. Show that if

$$2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u,$$

then QP is increasing throughout the flight of P.

(i) Notice that $P = \begin{pmatrix} u\cos\alpha t \\ u\sin\alpha t - \frac{1}{2}gt^2 \end{pmatrix}$, so

$$\begin{aligned} |OP|^2 &= u^2\cos^2\alpha t^2 + \left(u\sin\alpha t - \frac{1}{2}gt^2\right)^2 \\ &= u^2\cos^2\alpha t^2 + u^2\sin^2\alpha t^2 - u\sin\alpha g t^3 + \frac{1}{4}g^2t^4 \\ &= u^2t^2 - u\sin\alpha g t^3 + \frac{1}{4}g^2t^4 \\ \frac{\mathrm{d}|OP|^2}{\mathrm{d}t} &= 2u^2t - 3u\sin\alpha g t^2 + g^2t^3 \\ &= t\left(2u^2 - 3u\sin\alpha(gt) + (gt)^2\right) \\ \Delta &= 9u^2\sin^2\alpha - 4\cdot 2u^2 \cdot 1 \\ &= u^2(9\sin^2\alpha - 8) \end{aligned}$$

Therefore if $\sin\alpha < \frac{2\sqrt{2}}{3}$ the discriminant is negative, the quadratic factor is always positive and the distance $|OP|$ is always increasing.

Similarly, if $\sin\alpha > \frac{2\sqrt{2}}{3}$ then the derivative has a root. This means somewhere on its (possibly extended) trajectory OP is decreasing. This must be before it lands, since if it were after it 'landed' then both the x and y distances are increasing, therefore it cannot occur after it 'lands'.

(ii) Note that $Q = \begin{pmatrix} -vt \\ 0 \end{pmatrix}$

$$\begin{aligned} |QP|^2 &= (u\cos\alpha t + vt)^2 + \left(u\sin\alpha t - \frac{1}{2}gt^2\right)^2 \\ &= u^2\cos^2\alpha t^2 + 2u\cos\alpha v t^2 + v^2t^2 + u^2\sin^2\alpha t^2 - u\sin\alpha g t^3 + \frac{1}{4}g^2t^4 \\ &= (u^2 + 2uv\cos\alpha + v^2)t^2 - u\sin\alpha g t^3 + \frac{1}{4}g^2t^4 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \frac{\mathrm{d}|QP|^2}{\mathrm{d}t} &= 2(u^2 + uv\cos\alpha + v^2)t - 3u\sin\alpha g t^2 + g^2t^3 \\ &= t\left(2(u^2 + 2uv\cos\alpha + v^2) - 3u\sin\alpha(gt) + (gt)^2\right) \\ \Delta &= 9u^2\sin^2\alpha - 8(u^2 + 2uv\cos\alpha + v^2) \\ &= (9\sin^2\alpha - 8)u^2 - 16v\cos\alpha u - 8v^2 \\ &= \left((\sin\alpha - 2\sqrt{2}\cos\alpha)u - 2\sqrt{2}v\right)\left((\sin\alpha + 2\sqrt{2}\cos\alpha)u + 2\sqrt{2}v\right) \end{aligned}$$

Since the second bracket is clearly positive, the first bracket must be negative (for $\Delta < 0$ and our derivative to be positive), ie $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$