$A \sim 5 \cdot 5 + U[0, 10]$ $B \sim 4 \cdot 5 + 3\text{Po}(4)$ $C \sim 2 \cdot 5 + B(5, \frac{4}{5}) \cdot 5$

$$\begin{aligned}
\mathbb{P}(A \leq 27) &= \mathbb{P}(U \leq 2) = 0.2 \\
\mathbb{P}(B \leq 27) &= \mathbb{P}(3\text{Po}(4) \leq 7) \\
&= \mathbb{P}(Po(4) \leq 2) \\
&= e^{-4}(1 + 4 + \frac{4^2}{2}) \\
&= 0.23810\ldots \\
\mathbb{P}(C \leq 27) &= \mathbb{P}(5 \cdot B(5, \tfrac{4}{5}) \leq 17) \\
&= \mathbb{P}(B(5, \tfrac{4}{5}) \leq 3) \\
&= \binom{5}{0}(\tfrac{1}{5})^5 + \binom{5}{1}(\tfrac{4}{5})(\tfrac{1}{5})^4 + \binom{5}{2}(\tfrac{4}{5})^2(\tfrac{1}{5})^3 + \binom{5}{3}(\tfrac{4}{5})^3(\tfrac{1}{5})^2 + \\
&= 0.26272
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(\text{came via B}|\text{at least 27 minutes}) &= \frac{\mathbb{P}(\text{came via B and at least 27 minutes})}{\mathbb{P}(\text{at least 27 minutes})} \\
&= \frac{\frac{1}{3} \cdot 0.23810\ldots}{\frac{1}{3} \cdot 0.2 + \frac{1}{3} \cdot 0.23810\ldots + \frac{1}{3} \cdot 0.26272} \\
&= 0.3397\ldots \\
&= 0.340 \text{ (3 s.f.)}
\end{aligned}$$

(i)

$$\begin{aligned}
\mathbb{E}(A) &= 25 + 5 && = 30 \\
\mathbb{E}(B) &= 20 + 3 \cdot 4 && = 32 \\
\mathbb{E}(C) &= 10 + 5 \cdot 4 && = 30
\end{aligned}$$

A and C are equally good.

(ii)

$$\begin{aligned}
\mathbb{P}(A \leq 32) &= \mathbb{P}(U \leq 7) && = 0.7 \\
\mathbb{P}(B \leq 32) &= \mathbb{P}(Po(4) \leq 4) \\
&= e^{-4}(1 + 4 + 8 + \frac{4^3}{6}) && = 0.4334\ldots \\
\mathbb{P}(C \leq 32) &= \mathbb{P}(B(5, \tfrac{4}{5}) \leq 4) \\
&= 1 - \mathbb{P}(B(5, \tfrac{4}{5}) = 5) \\
&= 1 - \frac{4^5}{5^5} && = 0.67232
\end{aligned}$$

So you should choose route A.

Question (1992 STEP III Q16)

The probability that there are exactly n misprints in an issue of a newspaper is $\mathrm{e}^{-\lambda}\lambda^n/n!$ where λ is a positive constant. The probability that I spot a particular misprint is p, independent of what happens for other misprints, and $0 < p < 1$.

(i) If there are exactly $m+n$ misprints, what is the probability that I spot exactly m of them?

(ii) Show that, if I spot exactly m misprints, the probability that I have failed to spot exactly n misprints is

$$\frac{(1-p)^n\lambda^n}{n!}\mathrm{e}^{-(1-p)\lambda}.$$

(i) $\binom{m+n}{m}p^m(1-p)^n$

(ii)

$$\begin{aligned}
\mathbb{P}(\text{failed to spot } n \text{ misprints}|\text{spotted } m \text{ misprints}) &= \frac{\mathbb{P}(\text{failed to spot } n \text{ misprints and spotted } m \text{ misp}}{\mathbb{P}(\text{spotted } m \text{ misprints})} \\
&= \frac{\binom{m+n}{n}p^m(1-p)^n e^{-\lambda}\lambda^{m+n}/(n+m)!}{\sum_{k=0}^{\infty}\binom{m+k}{k}p^m(1-p)^k e^{-\lambda}\lambda^{m+k}/(n+k)!} \\
&= \frac{\binom{m+n}{n}(1-p)^n\lambda^n/(n+m)!}{\sum_{k=0}^{\infty}\binom{m+k}{k}(1-p)^k\lambda^k/(n+k)!} \\
&= \frac{(1-p)^n\lambda^n/n!}{\sum_{k=0}^{\infty}(1-p)^k\lambda^k/k!} \\
&= \frac{(1-p)^n\lambda^n}{n!}e^{-(1-p)\lambda}
\end{aligned}$$

Alternatively, given the missed misprints and spotted misprints are independent, we can view them as both following $Po(p\lambda)$ and $Po((1-p)\lambda)$ and so we obtain exactly this result, without calculation.

Question (1993 STEP II Q16)

At the terminus of a bus route, passengers arrive at an average rate of 4 per minute according to a Poisson process. Each minute, on the minute, one bus arrives with probability $\frac{1}{4}$, independently of the arrival of passengers or previous buses. Just after eight o'clock there is no-one at the bus stop.

(i) What is the probability that the first bus arrives at n minutes past 8?

(ii) If the first bus arrives at 8:05, what is the probability that there are m people waiting for it?

(iii) Each bus can take 25 people and, since it is the terminus, the bus arrive empty. Explain carefully how you would calculate, to two significant figures, the probability that when the first bus arrives it is unable to pick up all the passengers. Your method should need the use of a calculator and standard tables only. There is no need to carry out the calculation.

Question (1995 STEP I Q13)

A scientist is checking a sequence of microscope slides for cancerous cells, marking each cancerous cell that she detects with a red dye. The number of cancerous cells on a slide is random and has a Poisson distribution with mean μ. The probability that the scientist spots any one cancerous cell is p, and is independent of the probability that she spots any other one.

(i) Show that the number of cancerous cells which she marks on a single slide has a Poisson distribution of mean $p\mu$.

(ii) Show that the probability Q that the second cancerous cell which she marks is on the kth slide is given by

$$Q = \mathrm{e}^{-\mu p(k-1)} \left\{(1 + k\mu p)(1 - \mathrm{e}^{-\mu p}) - \mu p\right\}.$$

Question (1995 STEP II Q13)

Fly By Night Airlines run jumbo jets which seat N passengers. From long experience they know that a very small proportion ϵ of their passengers fail to turn up. They decide to sell $N+k$ tickets for each flight. If k is very small compared with N explain why they might expect

$$\mathrm{P}(r \text{ passengers fail to turn up}) = \frac{\lambda^r}{r!}\mathrm{e}^{-\lambda}$$

approximately, with $\lambda = N\epsilon$. For the rest of the question you may assume that the formula holds exactly.

Each ticket sold represents £A profit, but the airline must pay each passenger that it cannot fly £B where $B > A > 0$. Explain why, if r passengers fail to turn up, its profit, in pounds, is

$$A(N+k) - B\max(0, k-r),$$

where $\max(0, k-r)$ is the larger of 0 and $k-r$. Write down the expected profit u_k when $k = 0, 1, 2$ and 3. Find $v_k = u_{k+1} - u_k$ for general k and show that $v_k > v_{k+1}$. Show also that

$$v_k \to A - B$$

as $k \to \infty$.

Advise Fly By Night on how to choose k to maximise its expected profit u_k.

Question (1996 STEP III Q14)

Whenever I go cycling I start with my bike in good working order. However if all is well at time t, the probability that I get a puncture in the small interval $(t, t+\delta t)$ is $\alpha\,\delta t$. How many punctures can I expect to get on a journey during which my total cycling time is T?

When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time t, the repair will be completed in time $(t, t+\delta t)$ is $\beta\,\delta t$. If $p(t)$ is the probability that I am repairing a puncture at time t, write down an equation relating $p(t)$ to $p(t+\delta t)$, and derive from this a differential equation relating $p'(t)$ and $p(t)$. Show that

$$p(t) = \frac{\alpha}{\alpha+\beta}(1 - \mathrm{e}^{-(\alpha+\beta)t})$$

satisfies this differential equation with the appropriate initial condition.

Find an expression, involving α, β and T, for the time expected to be spent mending punctures during a journey of total time T. Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \qquad \text{if } (\alpha+\beta)T \text{ is small,}$$

and by

$$\frac{\alpha}{\alpha+\beta} \qquad \text{if } (\alpha+\beta)T \text{ is large.}$$

Question (1997 STEP II Q14)

Traffic enters a tunnel which is 9600 metres long, and in which overtaking is impossible. The number of vehicles which enter in any given time is governed by the Poisson distribution with mean 6 cars per minute. All vehicles travel at a constant speed until forced to slow down on catching up with a slower vehicle ahead. I enter the tunnel travelling at 30 m s^{-1} and all the other traffic is travelling at 32 m s^{-1}. What is the expected number of vehicles in the queue behind me when I leave the tunnel?

Assuming again that I travel at 30 m s^{-1}, but that all the other vehicles are independently equally likely to be travelling at 30 m s^{-1} or 32 m s^{-1}, find the probability that exactly two vehicles enter the tunnel within 20 seconds of my doing so and catch me up before I leave it. Find also the probability that there are exactly two vehicles queuing behind me when I leave the tunnel.
[Ignore the lengths of the vehicles.]

Question (1998 STEP II Q14)

The staff of Catastrophe College are paid a salary of A pounds per year. With a Teaching Assessment Exercise impending it is decided to try to lower the student failure rate by offering each lecturer an alternative salary of $B/(1+X)$ pounds, where X is the number of his or her students who fail the end of year examination. Dr Doom has N students, each with independent probability p of failure. Show that she should accept the new salary scheme if

$$A(N+1)p < B(1-(1-p)^{N+1}).$$

Under what circumstances could X, for Dr Doom, be modelled by a Poisson random variable? What would Dr Doom's expected salary be under this model?

$$\mathbb{E}[\text{salary}] = B\sum_{k=0}^{N} \frac{1}{1+k}\binom{N}{k}p^k(1-p)^{N-k}$$

$$\begin{aligned}
(q+x)^N &= \sum_{k=0}^{N}\binom{N}{k}x^k q^{N-k} \\
\Rightarrow \qquad \int_0^p (q+x)^N \mathrm{d}x &= \sum_{k=0}^{N}\binom{N}{k}\frac{p^{k+1}}{k+1}q^{N-k} \\
\frac{(q+p)^{N+1}-q^{N+1}}{N+1} &= \frac{p}{B}\mathbb{E}[\text{salary}] \\
\Rightarrow \qquad \mathbb{E}[\text{salary}] &= B\frac{1-q^{N+1}}{p(N+1)}
\end{aligned}$$

Therefore if $Ap(N+1) < B(1-(1-p)^{N+1})$ the expected value of the new salary is higher. (Whether or not the new salary is worth it in a risk adjusted sense is for the birds).

We could model X by a Poisson random variable if N is large and $Np = \lambda$ is small. Suppose $X \approx Po(\lambda)$ then

Question (1999 STEP III Q14)

In the basic version of Horizons (H1) the player has a maximum of n turns, where $n \geq 1$. At each turn, she has a probability p of success, where $0 < p < 1$. If her first success is at the rth turn, where $1 \leq r \leq n$, she collects r pounds and then withdraws from the game. Otherwise, her winnings are nil. Show that in H1, her expected winnings are

$$p^{-1}\left[1 + nq^{n+1} - (n+1)q^n\right] \quad \text{pounds,}$$

where $q = 1 - p$. The rules of H2 are the same as those of H1, except that n is randomly selected from a Poisson distribution with parameter λ. If $n = 0$ her winnings are nil. Otherwise she plays H1 with the selected n. Show that in H2, her expected winnings are

$$\frac{1}{p}\left(1 - \mathrm{e}^{-\lambda p}\right) - \lambda q \mathrm{e}^{-\lambda p} \quad \text{pounds.}$$

Question (2000 STEP III Q12)

In a lottery, any one of N numbers, where N is large, is chosen at random and independently for each player by machine. Each week there are $2N$ players and one winning number is drawn. Write down an exact expression for the probability that there are three or fewer winners in a week, given that you hold a winning ticket that week. Using the fact that

$$\left(1 - \frac{a}{n}\right)^n \approx \mathrm{e}^{-a}$$

for n much larger than a, or otherwise, show that this probability is approximately $\frac{2}{3}$.

Discuss briefly whether this probability would increase or decrease if the numbers were chosen by the players.

Show that the expected number of winners in a week, given that you hold a winning ticket that week, is $3 - N^{-1}$.

Question (2001 STEP I Q13)

Four students, one of whom is a mathematician, take turns at washing up over a long period of time. The number of plates broken by any student in this time obeys a Poisson distribution, the probability of any given student breaking n plates being $\mathrm{e}^{-\lambda}\lambda^n/n!$ for some fixed constant λ, independent of the number of breakages by other students. Given that five plates are broken, find the probability that three or more were broken by the mathematician.

Let X be the number of plates broken by the mathematician and Y by the other student. Then $X \sim Po(\lambda), Y \sim Po(3\lambda)$ and $X + Y \sim Po(4\lambda)$

$$\mathbb{P}(X = k | X + Y = n) = \frac{\mathbb{P}(X = k, Y = n - k)}{\mathbb{P}(X + Y = n)}$$

$$= \frac{e^{-\lambda}\lambda^k/k! \cdot e^{-3\lambda}(4\lambda)^{n-k}/(n-k)!}{e^{-4\lambda}(4\lambda)^n/n!}$$
$$= \binom{n}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{5}\right)^{n-k}$$

Therefore $X|X+Y=n \sim Binomial(n, \frac{1}{4})$

$$\mathbb{P}(X \geq 3|X+Y=n) = \binom{5}{3}\frac{3^2}{4^5} + \binom{5}{4}\frac{3}{4^5} + \binom{5}{5}\frac{1}{4^5}$$
$$= \frac{1}{4^5}(90+15+1)$$
$$= \frac{106}{4^5} = \frac{53}{512} \approx \frac{1}{10}$$

Question (2002 STEP II Q14)

A densely populated circular island is divided into N concentric regions R_1, R_2, …, R_N, such that the inner and outer radii of R_n are $n-1$ km and n km, respectively. The average number of road accidents that occur in any one day in R_n is $2 - n/N$, independently of the number of accidents in any other region. Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, X, that occur in it. Show that, in the long term, the average number of recorded accidents per day will be

$$2 - \frac{1}{6}\left(1+\frac{1}{N}\right)\left(4-\frac{1}{N}\right).$$

[Note: $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$.] Show also that

$$\mathbb{P}(X=k) = \frac{e^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N}(2n-1)(2N-n)^k e^{n/N}.$$

Suppose now that $N=3$ and that, on a particular day, two accidents were recorded. Show that the probability that R_2 had been selected is

$$\frac{48}{48+45e^{1/3}+25e^{-1/3}}.$$

The area of R_n is $\pi(n^2-(n-1)^2) = (2n-1)\pi$.
The area of the whole region is πN^2.

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\text{choose region } n]]$$
$$= \sum_{n=1}^{N}\left(2-\frac{n}{N}\right)\cdot\frac{(2n-1)\pi}{N^2\pi}$$
$$= \sum_{n=1}^{N}\left(2\cdot\frac{(2n-1)\pi}{N^2\pi} - \frac{n}{N}\cdot\frac{(2n-1)\pi}{N^2\pi}\right)$$

$$\begin{aligned}
&= 2 - \frac{1}{N^3}\sum_{n=1}^{N}(2n^2 - n) \\
&= 2 - \frac{1}{N^3}\left(\frac{2N(N+1)(2N+1)}{6} - \frac{N(N+1)}{2}\right) \\
&= 2 - \frac{N+1}{6N^2}\left(2(2N+1) - 3\right) \\
&= 2 - \frac{N+1}{6N^2}(4N-1) \\
&= 2 - \frac{1}{6}\left(1 + \frac{1}{N}\right)\left(4 - \frac{1}{N}\right)
\end{aligned}$$

Modelling each region as $Po(2 - n/N)$ we have

$$\begin{aligned}
\mathbb{P}(X = k) &= \sum_{n=1}^{N} \exp(-2 + n/N)\frac{(2 - n/N)^k}{k!}\frac{2n-1}{N^2} \\
&= \frac{e^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N} e^{n/N}(2N - n)^k(2n-1)
\end{aligned}$$

as desired.

Supposing $N = 3$ and two accidents then

$$\begin{aligned}
\mathbb{P}(R_2 | X = 2) &= \frac{\frac{3}{9}e^{-4/3}\frac{(\frac{4}{3})^2}{2!}}{\mathbb{P}(X = 2)} \\
&= \frac{\frac{3}{9}e^{-4/3}\frac{(\frac{4}{3})^2}{2!}}{\frac{1}{9}e^{-5/3}\frac{(\frac{5}{3})^2}{2!} + \frac{3}{9}e^{-4/3}\frac{(\frac{4}{3})^2}{2!} + \frac{5}{9}e^{-2/3}\frac{(\frac{3}{3})^2}{2!}} \\
&= \frac{3 \cdot 16}{25e^{-1/3} + 3 \cdot 16 + 5 \cdot 9e^{1/3}} \\
&= \frac{48}{25e^{-1/3} + 48 + 45e^{1/3}}
\end{aligned}$$

as required.

Question (2003 STEP I Q14)

Jane goes out with any of her friends who call, except that she never goes out with more than two friends in a day. The number of her friends who call on a given day follows a Poisson distribution with parameter 2. Show that the average number of friends she sees in a day is $2 - 4e^{-2}$. Now Jane has a new friend who calls on any given day with probability p. Her old friends call as before, independently of the new friend. She never goes out with more than two friends in a day. Find the average number of friends she now sees in a day.

Question (2003 STEP III Q12)

Brief interruptions to my work occur on average every ten minutes and the number of interruptions in any given time period has a Poisson distribution. Given that an interruption has just occurred, find the probability that I will have less than t minutes to work before the next interruption. If the random variable T is the time I have to work before the next interruption, find the probability density function of T.

I need an uninterrupted half hour to finish an important paper. Show that the expected number of interruptions before my first uninterrupted period of half an hour or more is $e^3 - 1$. Find also the expected length of time between interruptions that are less than half an hour apart. Hence write down the expected wait before my first uninterrupted period of half an hour or more.

Question (2005 STEP II Q13)

The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geq 2$) which contain fewer than two errors is denoted by Y. Show that $\mathbb{P}(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1+\lambda)e^{-\lambda}$ and $q = 1 - p$. Show also that, if λ is sufficiently small,

(i) $q \approx \frac{1}{2}\lambda^2$;

(ii) the largest value of n for which $\mathbb{P}(Y = n) \geq 1 - \lambda$ is approximately $2/\lambda$;

(iii) $\mathbb{P}(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$.

Question (2006 STEP I Q13)

A very generous shop-owner is hiding small diamonds in chocolate bars. Each diamond is hidden independently of any other diamond, and on average there is one diamond per kilogram of chocolate.

(i) I go to the shop and roll a fair six-sided die once. I decide that if I roll a score of N, I will buy $100N$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6}\left(\frac{1-\mathrm{e}^{-0.6}}{1-\mathrm{e}^{-0.1}}\right)$$

Show also that the expected number of diamonds I find is 0.35.

(ii) Instead, I decide to roll a fair six-sided die repeatedly until I score a 6. If I roll my first 6 on my Tth throw, I will buy $100T$ grams of chocolate. Show that the probability that I will have no diamonds is

$$\frac{\mathrm{e}^{-0.1}}{6-5\mathrm{e}^{-0.1}}$$

Calculate also the expected number of diamonds that I find. (You may find it useful to consider the the binomial expansion of $(1-x)^{-2}$.)

Not that the number of diamonds per kilogram is 1 so we are assuming it is $Po(M)$ where M is the mass in kg. In particular $\mathbb{E}[X] = M$ and $\mathbb{P}(X=0) = e^{-M}$

(i)

$$\begin{aligned}
\mathbb{P}(\text{no diamonds}) &= \sum_{n=1}^{6} \mathbb{P}(\text{no diamonds and roll } n) \\
&= \sum_{n=1}^{6} \tfrac{1}{6} e^{-\frac{n}{10}} \\
&= \frac{e^{-0.1}}{6}\left(\frac{1-e^{-0.6}}{1-e^{-0.1}}\right) \\
\mathbb{E}[\text{diamonds}] &= \sum_{n=1}^{6} \mathbb{E}(\text{diamonds}|N=n)\mathbb{P}(N=n) \\
&= \sum_{n=1}^{6} 0.1n \cdot \frac{1}{6} \\
&= 0.1 \cdot \frac{7}{2} = 0.35
\end{aligned}$$

(ii) $\mathbb{P}(T=k) = \left(\frac{5}{6}\right)^{t-1}\frac{1}{6}$, so

$$\begin{aligned}
\mathbb{P}(\text{no diamonds}) &= \sum_{n=1}^{\infty} \mathbb{P}(\text{no diamonds and } T=n) \\
&= \sum_{n=1}^{\infty} e^{-0.1n}\left(\frac{5}{6}\right)^{n-1}\frac{1}{6}
\end{aligned}$$

$$= \frac{e^{-0.1}}{6} \frac{1}{1 - \frac{5}{6}e^{-0.1}}$$
$$= \frac{e^{-0.1}}{6 - 5e^{-0.1}}$$

$$\begin{aligned}
\mathbb{E}[\text{diamonds}] &= \sum_{n=1}^{\infty} \mathbb{E}(\text{diamonds}|T = n)\mathbb{P}(T = n) \\
&= \sum_{n=1}^{\infty} 0.1n \cdot \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} \\
&= \frac{0.1}{6} \sum_{n=1}^{\infty} n \cdot \left(\frac{5}{6}\right)^{n-1} \\
&= \frac{1}{60} \frac{1}{(1 - \frac{5}{6})^2} \\
&= \frac{6}{10} = \frac{3}{5}
\end{aligned}$$

Question (2007 STEP I Q14)

The discrete random variable X has a Poisson distribution with mean λ.

(i) Sketch the graph $y = (x + 1)\,\mathrm{e}^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes. It is known that $\mathbb{P}(X \geq 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

(ii) It is known (instead) that $\mathbb{P}\,(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

(iii) It is known (instead) that $\mathbb{P}\,(X = 1 \,|\, X \leq 2) = r$, where r is a given number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).

Question (2010 STEP I Q13)

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation. The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p. Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Let X_t be the number of texts he recieves before t hours. So $X_t \sim P(t\lambda)$

$$\begin{aligned} \mathbb{P}(X_1 = 0 \cap X_2 > 0) &= e^{-\lambda} \cdot \left(1 - e^{-\lambda}\right) = p \\ \Rightarrow \quad e^{2\lambda} p &= e^{\lambda} - 1 \\ \Rightarrow \quad 0 &= pe^{2\lambda} - e^{\lambda} + 1 \\ \Rightarrow \quad e^{\lambda} &= \frac{1 \pm \sqrt{1-4p}}{2p} \end{aligned}$$

Which clearly has two positive roots if $4p < 1$. We need to show both roots are > 1. So considering the smaller one we are looking at:

$$\begin{aligned} & \frac{1-\sqrt{1-4p}}{2p} > 1 \\ \Leftrightarrow \quad & 1 - \sqrt{1-4p} > 2p \\ \Leftrightarrow \quad & 1 - 2p > \sqrt{1-4p} \\ \Leftrightarrow \quad & (1-2p)^2 > 1 - 4p \\ \Leftrightarrow \quad & 1 - 4p + 4p^2 > 1 - 4p \end{aligned}$$

which is clearly true.

We must have $e^{\lambda_1} \cdot e^{\lambda_2} = \frac{1}{p}$, so $\lambda_1 + \lambda_2 = -\ln p$ by considering the product of the roots in our quadratic. (Vieta).

Therefore the probability she waits between 1 and 2 hours in the morning is $e^{-(\lambda_1+\lambda_2)} \cdot (1 - e^{-(\lambda_1+\lambda_2)}) = p \cdot (1-p)$

Question (2012 STEP II Q13)

In this question, you may assume that $\int_0^\infty \mathrm{e}^{-x^2/2}\mathrm{d}x = \sqrt{\frac{1}{2}\pi}$. The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y. The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $\mathbb{P}(Y < y)$ and hence show that the probability density function of Y is $2\pi yk\mathrm{e}^{-\pi ky^2}$ for $y \geq 0$. Find $\mathbb{E}(Y)$ and show that $\mathrm{Var}(Y) = \dfrac{4-\pi}{4\pi k}$.

A circle radius y has a number of supermarkets X where $X \sim Po(k\pi y^2)$.

$$\mathbb{P}(X = 0) = e^{-k\pi y^2}\frac{1}{0!} = e^{-k\pi y^2}$$

The probability $\mathbb{P}(Y < y) = 1 - \mathbb{P}(Y \geq y) = 1 - e^{-k\pi y^2}$, and in particular $f_Y(y) = 2k\pi ye^{-k\pi y^2}$ (by differentiating).

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^\infty yf_Y(y)\mathrm{d}y \\
&= \int_0^\infty 2\pi y^2ke^{-\pi ky^2}\mathrm{d}y \\
\sigma^2 = \frac{1}{2k\pi}: \quad &= \pi k\sqrt{2\pi}\sigma\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}y^2e^{-\frac{1}{2}\cdot 2\pi ky^2}\mathrm{d}y \\
&= \pi k\sqrt{2\pi}\sigma\mathbb{E}\left(N(0,\sigma^2)^2\right) \\
&= \pi k\sqrt{2\pi}\sigma\sigma^2 \\
&= \pi k\sqrt{2\pi}\frac{1}{(2k\pi)^{3/2}} \\
&= \frac{1}{2\sqrt{k}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(Y^2) &= \int_0^\infty y^2f_Y(y)\mathrm{d}y \\
&= \int_0^\infty 2\pi y^3ke^{-\pi ky^2}\mathrm{d}y \\
&= \int_0^\infty y^2 2y\pi ke^{-\pi ky^2}\mathrm{d}y \\
&= \left[-y^2e^{-\pi ky^2}\right]_0^\infty + \int_0^\infty 2ye^{-\pi ky^2}\mathrm{d}y \\
&= \left[-\frac{1}{\pi k}e^{-\pi ky^2}\right]_0^\infty \\
&= \frac{1}{\pi k} \\
\Rightarrow \quad \mathrm{Var}(Y) &= \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 \\
&= \frac{1}{\pi k} - \frac{1}{4k}
\end{aligned}$$

$$= \frac{4-\pi}{4\pi k}$$

Question (2013 STEP II Q12)

The random variable U has a Poisson distribution with parameter λ. The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1,\ 3,\ 5,\ 7,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} U & \text{if } U \text{ is } 2,\ 4,\ 6,\ 8,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ in terms of λ, α and β, where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \quad \text{and} \quad \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots .$$

(ii) Show that

$$\text{Var}(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

and obtain the corresponding expression for $\text{Var}(Y)$. Are there any non-zero values of λ for which $\text{Var}(X) + \text{Var}(Y) = \text{Var}(X+Y)$?

(i)

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{r=1}^{\infty} r\mathbb{P}(X=r) \\
&= \sum_{j=1}^{\infty} (2j-1)\mathbb{P}(U=2j-1) \\
&= \sum_{j=1}^{\infty} (2j-1)\frac{e^{-\lambda}\lambda^{2j-1}}{(2j-1)!} \\
&= \sum_{j=1}^{\infty} e^{-\lambda}\frac{\lambda^{2j-1}}{(2j-2)!} \\
&= \lambda e^{-\lambda}\sum_{j=1}^{\infty} \frac{\lambda^{2j-2}}{(2j-2)!} \\
&= \lambda e^{-\lambda}\alpha
\end{aligned}$$

Since $\mathbb{E}(X+Y) = \lambda, \mathbb{E}(Y) = \lambda(1-e^{-\lambda}\alpha) = \lambda(e^{-\lambda}(\alpha+\beta) - e^{-\lambda}\alpha) = \lambda e^{-\lambda}\beta$. Alternatively, as $\beta + \alpha = e^{\lambda}$, $\mathbb{E}(X) = \frac{\lambda\alpha}{\alpha+\beta}, \mathbb{E}(Y) = \frac{\lambda\beta}{\alpha+\beta}$

(ii)

$$\text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$
\begin{aligned}
&= \sum_{odd} r^2 \mathbb{P}(U = r) - [\mathbb{E}(X)]^2 \\
&= \sum_{odd} (r(r-1) + r) \frac{e^{-\lambda}\lambda^r}{r!} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} \\
&= \sum_{odd} \frac{e^{-\lambda}\lambda^r}{(r-2)!} + \sum_{odd} \frac{e^{-\lambda}\lambda^r}{(r-1)!} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} \\
&= e^{-\lambda}\lambda^2\beta + e^{-\lambda}\lambda\alpha - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} \\
&= \frac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{Var}(Y) &= \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 \\
&= \sum_{even} r^2 \mathbb{P}(U = r) - [\mathbb{E}(Y)]^2 \\
&= \sum_{even} (r(r-1) + r) \frac{e^{-\lambda}\lambda^r}{r!} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} \\
&= e^{-\lambda}\lambda^2\alpha + e^{-\lambda}\lambda\beta - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} \\
&= \frac{\lambda\beta + \lambda^2\alpha}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2}
\end{aligned}
$$

Since $\mathrm{Var}(X+Y) = \mathrm{Var}(U) = \lambda$, we are interested in solving:

$$
\begin{aligned}
\lambda &= \frac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} + \frac{\lambda\beta + \lambda^2\alpha}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} \\
&= \frac{\lambda(\alpha+\beta) + \lambda^2(\alpha+\beta)}{\alpha+\beta} - \frac{\lambda^2(\alpha^2+\beta^2)}{(\alpha+\beta)^2} \\
&= \lambda + \lambda^2 \frac{(\alpha+\beta)^2 - (\alpha^2+\beta^2)}{(\alpha+\beta)^2} \\
&= \lambda + \lambda^2 \frac{2\alpha\beta}{(\alpha+\beta)^2}
\end{aligned}
$$

which is clearly not possible if $\lambda \neq 0$

Question (2015 STEP I Q12)

The number X of casualties arriving at a hospital each day follows a Poisson distribution with mean 8; that is,

$$\mathbb{P}(X = n) = \frac{e^{-8}8^n}{n!}, \qquad n = 0,\ 1,\ 2,\ \dots\ .$$

Casualties require surgery with probability $\frac{1}{4}$. The number of casualties arriving on any given day is independent of the number arriving on any other day and the casualties require surgery independently of one another.

(i) What is the probability that, on a day when exactly n casualties arrive, exactly r of them require surgery?

(ii) Prove (algebraically) that the number requiring surgery each day also follows a Poisson distribution, and state its mean.

(iii) Given that in a particular randomly chosen week a total of 12 casualties require surgery on Monday and Tuesday, what is the probability that 8 casualties require surgery on Monday? You should give your answer as a fraction in its lowest terms.

(i) $\mathbb{P}(r \text{ need surgery}|n \text{ casualties}) = \binom{n}{r}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}$

(ii)

$$\begin{aligned}
\mathbb{P}(r \text{ need surgery}) &= \sum_{n=r}^{\infty} \mathbb{P}(r \text{ need surgery}|n \text{ casualties})\mathbb{P}(n \text{ casualties}) \\
&= \sum_{n=r}^{\infty} \binom{n}{r}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}\frac{e^{-8}8^n}{n!} \\
&= \sum_{n=r}^{\infty} \frac{n!}{(n-r)!r!}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}\frac{e^{-8}8^n}{n!} \\
&= \frac{e^{-8}8^r}{r!}\left(\frac{1}{4}\right)^r\sum_{n=r}^{\infty}\frac{8^{n-r}}{(n-r)}\left(\frac{3}{4}\right)^{n-r} \\
&= \frac{e^{-8}8^r}{r!}\left(\frac{1}{4}\right)^r\sum_{n=r}^{\infty}\frac{6^{n-r}}{(n-r)} \\
&= \frac{e^{-8}2^r}{r!}e^6 \\
&= \frac{e^{-2}2^r}{r!}
\end{aligned}$$

Therefore the number requiring surgery is $Po(2)$ with mean 2.

(iii)

$$\mathbb{P}(X_1 = 8|X_1 + X_2 = 12) = \frac{\mathbb{P}(X_1 = 8, X_2 = 4)}{\mathbb{P}(X_1 + X_2 = 12)}$$

$$
\begin{aligned}
&= \frac{\frac{e^{-2}2^8}{8!} \cdot \frac{e^{-2}2^4}{4!}}{\frac{e^{-4}4^{12}}{12!}} \\
&= \frac{12!}{8!4!}\frac{1}{2^{12}} \\
&= \binom{12}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^8 \\
&= \frac{495}{4096}
\end{aligned}
$$

Question (2016 STEP III Q12)

Let X be a random variable with mean μ and standard deviation σ. *Chebyshev's inequality*, which you may use without proof, is

$$\mathbb{P}\left(|X-\mu| > k\sigma\right) \leq \frac{1}{k^2},$$

where k is any positive number.

(i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where n is an integer greater than 1. Let α be the probability that the coin lands heads up N times, where $16n \leq N \leq 24n$. Use Chebyshev's inequality to show that

$$\alpha \geq 1 - \frac{1}{n}.$$

(ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geq \left(1 - \frac{1}{n}\right)e^n.$$

(i) Let N be the number of times the coin lands heads up, ie $N \sim \mathit{Binomial}(100n, 0.2)$, then $\mathbb{E}(N) = \mu = 20n, \mathrm{Var}(N) = \sigma^2 = 100n \cdot 0.2 \cdot 0.8 = 16n \Rightarrow \sigma = 4\sqrt{n}$.

$$
\begin{aligned}
& & \mathbb{P}(|X-\mu| > k\sigma) &\leq \frac{1}{k^2} \\
&\Rightarrow & 1 - \mathbb{P}(|X-\mu| \leq k\sigma) &\leq \frac{1}{k^2} \\
&\Rightarrow & 1 - \mathbb{P}(|X-20n| \leq \sqrt{n}\cdot 4\sqrt{n}) &\leq \frac{1}{\sqrt{n}^2} \\
&\Rightarrow & 1 - \mathbb{P}(16n \leq N \leq 24n) &\leq \frac{1}{n} \\
&\Rightarrow & 1 - \frac{1}{n} &\leq \alpha
\end{aligned}
$$

(ii) Suppose $X \sim \mathit{Pois}(n)$, then $\mathbb{E}(X) = n, \mathrm{Var}(X) = n$. Therefore

$$\mathbb{P}(|X-\mu| > k\sigma) \leq \frac{1}{k^2}$$

$$\Rightarrow \qquad 1 - \mathbb{P}(|X - n| \leq \sqrt{n} \cdot \sqrt{n}) > \frac{1}{\sqrt{n}^2}$$

$$\Rightarrow \qquad 1 - \sum_{i=0}^{2n} \mathbb{P}(X = i) \leq \frac{1}{n}$$

$$\Rightarrow \qquad \sum_{i=0}^{2n} e^{-n} \frac{n^i}{i!} \geq 1 - \frac{1}{n}$$

$$\Rightarrow \qquad \sum_{i=0}^{2n} \frac{n^i}{i!} \geq \left(1 - \frac{1}{n}\right) e^n$$

Question (2017 STEP II Q12)

Adam and Eve are catching fish. The number of fish, X, that Adam catches in any time interval is Poisson distributed with parameter λt, where λ is a constant and t is the length of the time interval. The number of fish, Y, that Eve catches in any time interval is Poisson distributed with parameter μt, where μ is a constant and t is the length of the time interval

The two Poisson variables are independent. You may assume that the expected time between Adam catching a fish and Adam catching his next fish is λ^{-1}, and similarly for Eve.

(i) By considering $\mathbb{P}(X + Y = r)$, show that the total number of fish caught by Adam and Eve in time T also has a Poisson distribution.

(ii) Given that Adam and Eve catch a total of k fish in time T, where k is fixed, show that the number caught by Adam has a binomial distribution.

(iii) Given that Adam and Eve start fishing at the same time, find the probability that the first fish is caught by Adam.

(iv) Find the expected time from the moment Adam and Eve start fishing until they have each caught at least one fish.

[**Note** This question has been redrafted to make the meaning clearer.]

(i)

$$\begin{aligned}
\mathbb{P}(X + Y = r) &= \sum_{k=0}^{r} \mathbb{P}(X = k, Y = r - k) \\
&= \sum_{k=0}^{r} \mathbb{P}(X = k)\mathbb{P}(Y = r - k) \\
&= \sum_{k=0}^{r} \frac{e^{-\lambda T}(\lambda T)^k}{k!} \frac{e^{-\mu T}(\mu T)^{r-k}}{(r-k)!} \\
&= \frac{e^{-(\mu+\lambda)T}}{r!} \sum_{k=0}^{r} \binom{r}{k} (\lambda T)^k (\mu T)^{r-k}
\end{aligned}$$

$$= \frac{e^{-(\mu+\lambda)T}((\mu+\lambda)T)^r}{r!}$$

Therefore $X + Y \sim Po\,((\mu+\lambda)T)$

(ii)

$$\begin{aligned}\mathbb{P}(X = r|X+Y=k) &= \frac{\mathbb{P}(X=r, Y=k-r)}{\mathbb{P}(X+Y=k)} \\ &= \frac{\frac{e^{-\lambda T}(\lambda T)^r}{r!}\frac{e^{-\mu T}(\mu T)^{k-r}}{(k-r)!}}{\frac{e^{-(\mu+\lambda)T}((\mu+\lambda)T)^k}{k!}} \\ &= \binom{k}{r}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{k-r}\end{aligned}$$

Therefore $X|X+Y=k \sim B(k, \frac{\lambda}{\lambda+\mu})$

(iii) $P(X = 1|X+Y=1) = \frac{\lambda}{\lambda+\mu}$

(iv) Let X_1, Y_1 be the time to the first fish are caught by Adam and Eve, then

$$\begin{aligned}\mathbb{P}(X_1, Y_1 > t) &= \mathbb{P}(X_1 > t)\mathbb{P}(Y_1 > t) \\ &= e^{-\lambda t}e^{-\mu t} \\ &= e^{-(\lambda+\mu)t} \\ \Rightarrow \qquad f_{\max(X_1,Y_1)}(t) &= (\lambda+\mu)e^{-(\lambda+\mu)}\end{aligned}$$

Therefore the expected time is $\frac{1}{\mu+\lambda}$

Question (2019 STEP III Q11)

The number of customers arriving at a builders' merchants each day follows a Poisson distribution with mean λ. Each customer is offered some free sand. The probability of any given customer taking the free sand is p.

(i) Show that the number of customers each day who take sand follows a Poisson distribution with mean $p\lambda$.

(ii) The merchant has a mass S of sand at the beginning of the day. Each customer who takes the free sand gets a proportion k of the remaining sand, where $0 \leq k < 1$. Show that by the end of the day the expected mass of sand taken is

$$\left(1 - e^{-kp\lambda}\right) S.$$

(iii) At the beginning of the day, the merchant's bag of sand contains a large number of grains, exactly one of which is made from solid gold. At the end of the day, the merchant's assistant takes a proportion k of the remaining sand. Find the probability that the assistant takes the golden grain. Comment on the case $k = 0$ and on the limit $k \to 1$. In the case $p\lambda > 1$ find the value of k which maximises the probability that the assistant takes the golden grain.

(i) Let X be the number of people arriving on a given day, and Y be the number taking sand, then

$$\begin{aligned}
\mathbb{P}(Y = k) &= \sum_{x=k}^{\infty} \mathbb{P}(x \text{ arrive and } k \text{ of them take sand}) \\
&= \sum_{x=k}^{\infty} \mathbb{P}(X = x)\mathbb{P}(k \text{ out of } x \text{ of them take sand}) \\
&= \sum_{x=k}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} \binom{x}{k} p^k (1-p)^{x-k} \\
&= e^{-\lambda} \left(\frac{p}{1-p}\right)^k \sum_{x=k}^{\infty} \frac{((1-p)\lambda)^x}{k!(x-k)!} \\
&= e^{-\lambda} \left(\frac{p}{1-p}\right)^k \frac{((1-p)\lambda)^k}{k!} \sum_{x=0}^{\infty} \frac{((1-p)\lambda)^x}{x!} \\
&= e^{-\lambda} \left(\frac{p}{1-p}\right)^k \frac{((1-p)\lambda)^k}{k!} e^{(1-p)\lambda} \\
&= e^{-p\lambda} \frac{(p\lambda)^k}{k!}
\end{aligned}$$

which is precisely a Poisson with parameter $p\lambda$.

Alternatively, $Y = B_1 + B_2 + \cdots + B_X$ where $B_i \sim \mathit{Bernoulli}(p)$ so $G_Y(t) = G_X(G_B(t)) = G_X(1 - p + pt) = e^{-\lambda(1-(1-p+pt))} = e^{-p\lambda(1-t)}$ so $Y \sim Po(\lambda)$

Alternatively, alternatively, let Z be the number of people not taking sand, so

$$\begin{aligned}\mathbb{P}(Y=y, Z=z) &= \mathbb{P}(X=y+z)\cdot\binom{y+z}{y}p^y(1-p)^z \\ &= e^{-\lambda}\frac{\lambda^{y+z}}{(y+z)!}\frac{(y+z)!}{y!z!}p^y(1-p)^z \\ &= \left(e^{-p\lambda}\frac{(p\lambda)^y}{y!}\right)\cdot\left(e^{-(1-p)\lambda}\frac{((1-p)\lambda)^z}{z!}\right)\end{aligned}$$

So clearly Y and Z are both (independent!) Poisson with parameters $p\lambda$ and $(1-p)\lambda$

(ii) The amount taken is $Sk + S(1-k)k + \cdots + Sk(1-k)^{Y-1} = Sk\cdot\frac{1-(1-k)^Y}{k} = S(1-(1-k)^Y)$ so

$$\begin{aligned}\mathbb{E}[\text{taken sand}] &= \mathbb{E}\left[S(1-(1-k)^Y)\right] \\ &= S - S\mathbb{E}\left[(1-k)^Y\right] \\ &= S - SG_Y(1-k) \\ &= S - Se^{-p\lambda(1-(1-k))} && \text{(pgf for Poisson)} \\ &= S\left(1-e^{-kp\lambda}\right)\end{aligned}$$

(iii) The fraction of grains the assistant takes home is:

$(1-k)^Y k$, which has expected value $ke^{-kp\lambda}$. This the the probability he takes home the golden grain.

When $k=0$ the probability is 0 which makes sense (no-one takes home any sand, including the merchant's assistant).

As $k \to 1$ we get $e^{-p\lambda}$ which is the probability that no-one gets any sand other than him.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\left(ke^{-kp\lambda}\right) &= e^{-kp\lambda} - (p\lambda)ke^{-kp\lambda} \\ &= e^{-kp\lambda}(1-(p\lambda)k)\end{aligned}$$

Therefore maximised at $k = \frac{1}{p\lambda}$. (Clearly this is a maximum just by sketching the function)

Question (2025 STEP II Q12)

Let X be a Poisson random variable with mean λ and let $p_r = P(X = r)$, for $r = 0, 1, 2, \ldots$. Neither λ nor $\lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}$ is an integer.

(i) Show, by considering the sequence $d_r \equiv p_r - p_{r-1}$ for $r = 1, 2, \ldots$, that there is a unique integer m such that $P(X = r) \leq P(X = m)$ for all $r = 0, 1, 2, \ldots$, and that

$$\lambda - 1 < m < \lambda.$$

(ii) Show that the minimum value of d_r occurs at $r = k$, where k is such that

$$k < \lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}} < k + 1.$$

(iii) Show that the condition for the maximum value of d_r to occur at $r = 1$ is

$$1 < \lambda < 2 + \sqrt{2}.$$

(iv) In the case $\lambda = 3.36$, sketch a graph of p_r against r for $r = 0, 1, 2, \ldots, 6, 7$.

(i) Suppose $d_r = p_r - p_{r-1}$ then

$$\begin{aligned} d_r &= p_r - p_{r-1} \\ &= \mathbb{P}(X = r) - \mathbb{P}(X = r - 1) \\ &= e^{-\lambda}\left(\frac{\lambda^r}{r!} - \frac{\lambda^{r-1}}{(r-1)!}\right) \\ &= e^{-\lambda}\frac{\lambda^{r-1}}{(r-1)!}\left(\frac{\lambda}{r} - 1\right) \end{aligned}$$

Therefore $d_r > 0 \Leftrightarrow \lambda > r$ie, p_r is increasing while $r < \lambda$ and reaches a (unique) maximum when $r = \lfloor \lambda \rfloor$.

(ii) Let $dd_r = d_r - d_{r-1}$, so:

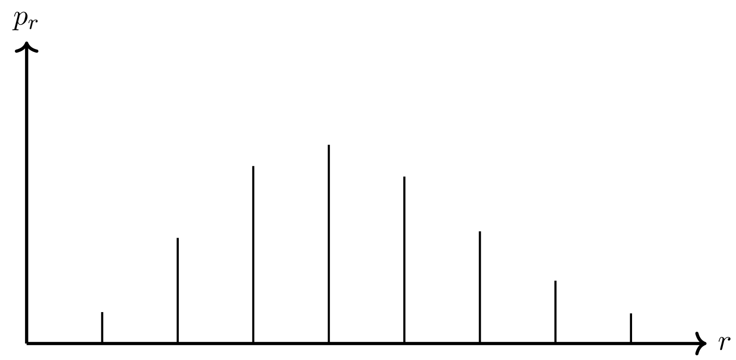
$$\begin{aligned} dd_r &= d_r - d_{r-1} \\ &= p_r - 2p_{r-1} + p_{r-2} \\ &= e^{-\lambda}\frac{\lambda^{r-2}}{r!}\left(\lambda^2 - 2\lambda r + r(r-1)\right) \end{aligned}$$

Therefore $dd_r < 0 \Leftrightarrow \lambda^2 - 2\lambda r + r(r-1) < 0 \Leftrightarrow r^2 - (1+2\lambda)r + \lambda^2 < 0$, but this has roots $r = \frac{(1+2\lambda)\pm\sqrt{(1+2\lambda)^2-4\lambda^2}}{2} = \lambda + \frac{1}{2} \pm \sqrt{\lambda + \frac{1}{4}}$. Therefore d_r is decreasing when $r \in \left(\lambda + \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}, \lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}\right)$, therefore the possible minimums are d_1 and d_k where $k < \lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}} < k + 1$. $d_1 = e^{-\lambda}(\lambda - 1)$, $d_k = e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}(\frac{\lambda}{k} - 1)$

(iii) If the maximum value of d_r is $r = 1$ then d_r must be decreasing, ie considering dd_2 we have $\lambda^2 - 4\lambda + 2 < 0 \Leftrightarrow 2 - \sqrt{2} < \lambda < 2 + \sqrt{2}$. It must also be the case that it doesn't get beaten as $\lambda \to \infty$. In this case $d_r \to 0$, so we need $d_1 > 0$, ie $\lambda > 1$.

Therefore $1 < \lambda < 2 + \sqrt{2}$

(iv)



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