

**Question (1989 STEP II Q10)**

State carefully the conditions which the fixed vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{u}$  and  $\mathbf{v}$  must satisfy in order to ensure that the line  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$  intersects the line  $\mathbf{r} = \mathbf{b} + \mu\mathbf{v}$  in exactly one point. Find the two values of the fixed scalar  $b$  for which the planes with equations

$$\left. \begin{aligned} x + y + bz &= b + 2 \\ bx + by + z &= 2b + 1 \end{aligned} \right\} \quad (*)$$

do not intersect in a line. For other values of  $b$ , express the line of intersection of the two planes in the form  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ , where  $\mathbf{a} \cdot \mathbf{u} = 0$ . Find the conditions which  $b$  and the fixed scalars  $c$  and  $d$  must satisfy to ensure that there is exactly one point on the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \mu \begin{pmatrix} 1 \\ d \\ 0 \end{pmatrix}$$

whose coordinates satisfy both equations (\*).

**Question (1991 STEP III Q2)**

The distinct points  $P_1, P_2, P_3, Q_1, Q_2$  and  $Q_3$  in the Argand diagram are represented by the complex numbers  $z_1, z_2, z_3, w_1, w_2$  and  $w_3$  respectively. Show that the triangles  $P_1P_2P_3$  and  $Q_1Q_2Q_3$  are similar, with  $P_i$  corresponding to  $Q_i$  ( $i = 1, 2, 3$ ) and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1 - z_2}{z_2 - z_3} = \frac{w_1 - w_2}{w_1 - w_3}.$$

Verify that this condition may be written

$$\det \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

- (i) Show that if  $w_i = z_i^2$  ( $i = 1, 2, 3$ ) then triangle  $P_1P_2P_3$  is not similar to triangle  $Q_1Q_2Q_3$ .
- (ii) Show that if  $w_i = z_i^3$  ( $i = 1, 2, 3$ ) then triangle  $P_1P_2P_3$  is similar to triangle  $Q_1Q_2Q_3$  if and only if the centroid of triangle  $P_1P_2P_3$  is the origin. [The *centroid* of triangle  $P_1P_2P_3$  is represented by the complex number  $\frac{1}{3}(z_1 + z_2 + z_3)$ .]
- (iii) Show that the triangle  $P_1P_2P_3$  is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

**Question (1991 STEP III Q4)**

The point  $P$  moves on a straight line in three-dimensional space. The position of  $P$  is observed from the points  $O_1(0, 0, 0)$  and  $O_2(8a, 0, 0)$ . At times  $t = t_1$  and  $t = t'_1$ , the lines of sight from  $O_1$  are along the lines

$$\frac{x}{2} = \frac{z}{3}, y = 0 \quad \text{and} \quad x = 0, \frac{y}{3} = \frac{z}{4}$$

respectively. At times  $t = t_2$  and  $t = t'_2$ , the lines of sight from  $O_2$  are

$$\frac{x - 8a}{-3} = \frac{y}{1} = \frac{z}{3} \quad \text{and} \quad \frac{x - 8a}{-4} = \frac{y}{2} = \frac{z}{5}$$

respectively. Find an equation or equations for the path of  $P$ .

**Question (1993 STEP III Q3)**

The matrices  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{M}$  are given by

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 13 & 5 \\ 3 & 8 & 7 \end{pmatrix},$$

where  $a, b, \dots, r$  are real numbers. Given that  $\mathbf{M} = \mathbf{AB}$ , show that  $a = 1, b = 4, c = 1, d = 3, e = 1, f = -2, p = 3, q = 2$  and  $r = -3$  gives the *unique* solution for  $\mathbf{A}$  and  $\mathbf{B}$ . Evaluate  $\mathbf{A}^{-1}$  and  $\mathbf{B}^{-1}$ ,

Hence, or otherwise, solve the simultaneous equations

$$\begin{aligned} x + 3y + 2z &= 7 \\ 4x + 13y + 5z &= 18 \\ 3x + 8y + 7z &= 25. \end{aligned}$$

**Question (1995 STEP III Q1)**

Find the simultaneous solutions of the three linear equations

$$\begin{aligned} a^2x + ay + z &= a^2 \\ ax + y + bz &= 1 \\ a^2bx + y + bz &= b \end{aligned}$$

for all possible real values of  $a$  and  $b$ .

**Question (1996 STEP II Q2)**

Consider the system of equations

$$2yz + zx - 5xy = 2$$

$$yz - zx + 2xy = 1$$

$$yz - 2zx + 6xy = 3$$

Show that

$$xyz = \pm 6$$

and find the possible values of  $x$ ,  $y$  and  $z$ .

**Question (1996 STEP III Q2)**

For all values of  $a$  and  $b$ , either solve the simultaneous equations

$$x + y + az = 2$$

$$x + ay + z = 2$$

$$2x + y + z = 2b$$

or prove that they have no solution.

**Question (2003 STEP II Q1)**

Consider the equations

$$ax - y - z = 3,$$

$$2ax - y - 3z = 7,$$

$$3ax - y - 5z = b,$$

where  $a$  and  $b$  are given constants.

- (i) In the case  $a = 0$ , show that the equations have a solution if and only if  $b = 11$ .
- (ii) In the case  $a \neq 0$  and  $b = 11$  show that the equations have a solution with  $z = \lambda$  for any given number  $\lambda$ .
- (iii) In the case  $a = 2$  and  $b = 11$  find the solution for which  $x^2 + y^2 + z^2$  is least.
- (iv) Find a value for  $a$  for which there is a solution such that  $x > 10^6$  and  $y^2 + z^2 < 1$ .

**Question (2003 STEP II Q5)**

The position vectors of the points  $A$ ,  $B$  and  $P$  with respect to an origin  $O$  are  $a\mathbf{i}$ ,  $b\mathbf{j}$  and  $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ , respectively, where  $a$ ,  $b$ , and  $n$  are all non-zero. The points  $E$ ,  $F$ ,  $G$  and  $H$  are the midpoints of  $OA$ ,  $BP$ ,  $OB$  and  $AP$ , respectively. Show that the lines  $EF$  and  $GH$  intersect.

Let  $D$  be the point with position vector  $d\mathbf{k}$ , where  $d$  is non-zero, and let  $S$  be the point of intersection of  $EF$  and  $GH$ . The point  $T$  is such that the mid-point of  $DT$  is  $S$ . Find the position vector of  $T$  and hence find  $d$  in terms of  $n$  if  $T$  lies in the plane  $OAB$ .

**Question (2004 STEP II Q6)**

The vectors  $\mathbf{a}$  and  $\mathbf{b}$  lie in the plane  $\Pi$ . Given that  $|\mathbf{a}| = 1$  and  $\mathbf{a} \cdot \mathbf{b} = 3$ , find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , a vector  $\mathbf{p}$  parallel to  $\mathbf{a}$  and a vector  $\mathbf{q}$  perpendicular to  $\mathbf{a}$ , both lying in the plane  $\Pi$ , such that

$$\mathbf{p} + \mathbf{q} = \mathbf{a} + \mathbf{b} .$$

The vector  $\mathbf{c}$  is not parallel to the plane  $\Pi$  and is such that  $\mathbf{a} \cdot \mathbf{c} = -2$  and  $\mathbf{b} \cdot \mathbf{c} = 2$ . Given that  $|\mathbf{b}| = 5$ , find, in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ , vectors  $\mathbf{P}$ ,  $\mathbf{Q}$  and  $\mathbf{R}$  such that  $\mathbf{P}$  and  $\mathbf{Q}$  are parallel to  $\mathbf{p}$  and  $\mathbf{q}$ , respectively,  $\mathbf{R}$  is perpendicular to the plane  $\Pi$  and

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c} .$$