

Question (1988 STEP III Q2)

The real numbers u_0, u_1, u_2, \dots satisfy the difference equation

$$au_{n+2} + bu_{n+1} + cu_n = 0 \quad (n = 0, 1, 2, \dots),$$

where a, b and c are real numbers such that the quadratic equation

$$ax^2 + bx + c = 0$$

has two distinct real roots α and β . Show that the above difference equation is satisfied by the numbers u_n defined by

$$u_n = A\alpha^n + B\beta^n,$$

where

$$A = \frac{u_1 - \beta u_0}{\alpha - \beta} \quad \text{and} \quad B = \frac{u_1 - \alpha u_0}{\beta - \alpha}.$$

Show also, by induction, that these numbers provide the only solution. Find the numbers v_n ($n = 0, 1, 2, \dots$) which satisfy

$$8(n+2)(n+1)v_{n+2} - 2(n+3)(n+1)v_{n+1} - (n+3)(n+2)v_n = 0$$

with $v_0 = 0$ and $v_1 = 1$.

Question (1988 STEP III Q15)

Each day, books returned to a library are placed on a shelf in order of arrival, and left there. When a book arrives for which there is no room on the shelf, that book and all books subsequently returned are put on a trolley. At the end of each day, the shelf and trolley are cleared. There are just two-sizes of book: thick, requiring two units of shelf space; and thin, requiring one unit. The probability that a returned book is thick is p , and the probability that it is thin is $q = 1 - p$. Let $M(n)$ be the expected number of books that will be put on the shelf, when the length of the shelf is n units and n is an integer, on the assumption that more books will be returned each day than can be placed on the shelf. Show, giving reasoning, that

(i) $M(0) = 0$;

(ii) $M(1) = q$;

(iii) $M(n) - qM(n-1) - pM(n-2) = 1$, for $n \geq 2$.

Verify that a possible solution to these equations is

$$M(n) = A(-p)^n + B + Cn,$$

where A, B and C are numbers independent of n which you should express in terms of p .

Question (1998 STEP III Q3)

The value V_N of a bond after N days is determined by the equation

$$V_{N+1} = (1 + c)V_N - d \quad (c > 0, d > 0),$$

where c and d are given constants. By looking for solutions of the form $V_T = Ak^T + B$ for some constants A, B and k , or otherwise, find V_N in terms of V_0 . What is the solution for $c = 0$? Show that this is the limit (for fixed N) as $c \rightarrow 0$ of your solution for $c > 0$.

Question (2000 STEP III Q8)

The sequence a_n is defined by $a_0 = 1$, $a_1 = 1$, and

$$a_n = \frac{1 + a_{n-1}^2}{a_{n-2}} \quad (n \geq 2).$$

Prove by induction that

$$a_n = 3a_{n-1} - a_{n-2} \quad (n \geq 2).$$

Hence show that

$$a_n = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}} \quad (n \geq 1),$$

where $\alpha = \frac{1 + \sqrt{5}}{2}$.

Question (2008 STEP III Q8) (i) The coefficients in the series

$$S = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \cdots + a_r x^r + \cdots$$

satisfy a recurrence relation of the form $a_{r+1} + pa_r = 0$. Write down the value of p . By considering $(1 + px)S$, find an expression for the sum to infinity of S (assuming that it exists). Find also an expression for the sum of the first $n + 1$ terms of S .

(ii) The coefficients in the series

$$T = 2 + 8x + 18x^2 + 37x^3 + \cdots + a_r x^r + \cdots$$

satisfy a recurrence relation of the form $a_{r+2} + pa_{r+1} + qa_r = 0$. Find an expression for the sum to infinity of T (assuming that it exists). By expressing T in partial fractions, or otherwise, find an expression for the sum of the first $n + 1$ terms of T .