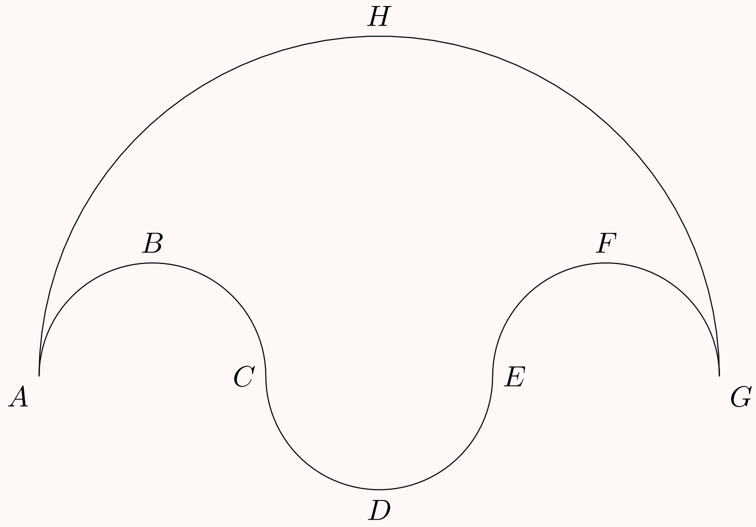
Question (1988 STEP I Q3)

Two points P and Q lie within, or on the boundary of, a square of side 1cm, one corner of which is the point O. Show that the length of at least one of the lines OP, PQ and QO must be less than or equal to $(\sqrt{6}-\sqrt{2})$ cm.

Question (1989 STEP I Q1)



In the above diagram, ABC, CDE, EFG and AHG are semicircles and A, C, E, G lie on a straight line. The radii of ABC, EFG, AHG are h, h and r respectively, where $2h < r$. Show that the area enclosed by $ABCDEFGH$ is equal to that of a circle with diameter HD. Each semicircle is now replaced by a portion of a parabola, with vertex at the midpoint of the semicircle arc, passing through the endpoints (so, for example, ABC is replaced by part of a parabola passing through A and C and with vertex at B). Find a formula in terms of r and h for the area enclosed by $ABCDEFGH$.

Question (1989 STEP III Q1)

Prove that the area of the zone of the surface of a sphere between two parallel planes cutting the sphere is given by

$$2\pi \times (\text{radius of sphere}) \times (\text{perpendicular distance between the planes}).$$

A tangent from the origin O to the curve with cartesian equation

$$(x-c)^2 + y^2 = a^2,$$

where a and c are positive constants with $c > a$, touches the curve at P. The x-axis cuts the curve at Q and R, the points lying in the order OQR on the axis. The line OP and the arc PR are rotated through 2π radians about the line OQR to form a surface. Find the area of this surface.

Question (1993 STEP I Q8) **(i)** Prove that the intersection of the surface of a sphere with a plane is always a circle, a point or the empty set. Prove that the intersection of the surfaces of two spheres with distinct centres is always a circle, a point or the empty set.

[If you use coordinate geometry, a careful choice of origin and axes may help.]

(ii) The parish council of Little Fitton have just bought a modern sculpture entitled 'Truth, Love and Justice pouring forth their blessings on Little Fitton.' It consists of three vertical poles AD, BE and CF of heights 2 metres, 3 metres and 4 metres respectively. Show that $\angle DEF = \cos^{-1}\frac{1}{5}$.

Vandals now shift the pole AD so that A is unchanged and the pole is still straight but D is vertically above AB with $\angle BAD = \frac{1}{4}\pi$ (in radians). Find the new angle $\angle DEF$ in radians correct to four figures.

Question (1994 STEP I Q1)

My house has an attic consisting of a horizontal rectangular base of length $2q$ and breadth $2p$ (where $p < q$) and four plane roof sections each at angle θ to the horizontal. Show that the length of the roof ridge is independent of θ and find the volume of the attic and the surface area of the roof.

Question (1994 STEP II Q7)

Show that the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0$$

where $a > 0$ and $f^2 + g^2 > ac$ represents a circle in Cartesian coordinates and find its centre.

The smooth and level parade ground of the First Ruritanian Infantry Division is ornamented by two tall vertical flagpoles of heights h_1 and h_2 a distance d apart. As part of an initiative test a soldier has to march in such a way that he keeps the angles of elevation of the tops of the two flagpoles equal to one another. Show that if the two flagpoles are of different heights he will march in a circle. What happens if the two flagpoles have the same height?

To celebrate the King's birthday a third flagpole is added. Soldiers are then assigned to each of the three different pairs of flagpoles and are told to march in such a way that they always keep the tops of their two assigned flagpoles at equal angles of elevation to one another. Show that, if the three flagpoles have different heights h_1, h_2 and h_3 and the circles in which the soldiers march have centres of (x_{ij}, y_{ij}) (for the flagpoles of height h_i and h_j) relative to Cartesian coordinates fixed in the parade ground, then the x_{ij} satisfy

$$h_3^2\left(h_1^2 - h_2^2\right)x_{12} + h_1^2\left(h_2^2 - h_3^2\right)x_{23} + h_2^2\left(h_3^2 - h_1^2\right)x_{31} = 0,$$

and the same equation connects the y_{ij}. Deduce that the three centres lie in a straight line.

Question (1995 STEP II Q5)

The famous film star Birkhoff Maclane is sunning herself by the side of her enormous circular swimming pool (with centre O) at a point A on its circumference. She wants a drink from a small jug of iced tea placed at the diametrically opposite point B. She has three choices:

(i) to swim directly to B.

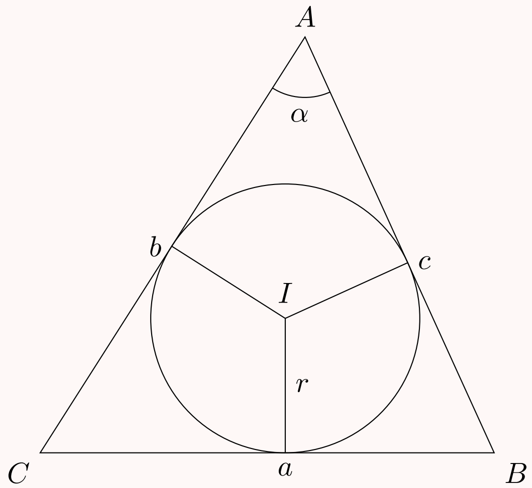
(ii) to choose θ with $0 < \theta < \pi$, to run round the pool to a point X with $\angle AOX = \theta$ and then to swim directly from X to B.

(iii) to run round the pool from A to B.

She can run k times as fast as she can swim and she wishes to reach her tea as fast as possible. Explain, with reasons, which of **(i)**, **(ii)** and **(iii)** she should choose for each value of k. Is there one choice from **(i)**, **(ii)** and **(iii)** she will never take whatever the value of k?

Question (1995 STEP II Q7)

The diagram shows a circle, of radius r and centre I, touching the three sides of a triangle ABC. We write a for the length of BC and α for the angle $\angle BAC$ and so on. Let $s = \frac{1}{2}(a + b + c)$ and let $\triangle$ be the area of the triangle.



(i) By considering the area of the triangles AIB, BIC and CIA, or otherwise, show that $\Delta = rs$.

(ii) By using the formula $\Delta = \frac{1}{2}bc\sin\alpha$, show that

$$\Delta^2 = \tfrac{1}{16}[4b^2c^2 - (2bc\cos\alpha)^2].$$

Now use the formula $a^2 = b^2 + c^2 - 2bc\cos\alpha$ to show that

$$\Delta^2 = \tfrac{1}{16}[(a^2 - (b-c)^2)((b+c)^2 - a^2)]$$

and deduce that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

(iii) A hole in the shape of the triangle ABC is cut in the top of a level table. A sphere of radius R rests in the hole. Find the height of the centre of the sphere above the level of the table top, expressing your answer in terms of a, b, c, s and R.

Question (1996 STEP II Q7)

Consider a fixed square $ABCD$ and a variable point P in the plane of the square. We write the perpendicular distance from P to AB as p, from P to BC as q, from P to CD as r and from P to DA as s. (Remember that distance is never negative, so $p, q, r, s \geqslant 0$.) If $pr = qs$, show that the only possible positions of P lie on two straight lines and a circle and that every point on these two lines and a circle is indeed a possible position of P.

Question (1999 STEP I Q2)

A point moves in the x-y plane so that the sum of the squares of its distances from the three fixed points (x_1, y_1), (x_2, y_2), and (x_3, y_3) is always a^2. Find the equation of the locus of the point and interpret it geometrically. Explain why a^2 cannot be less than the sum of the squares of the distances of the three points from their centroid. [The *centroid* has coordinates $(\bar{x}, \bar{y})$ where $3\bar{x} = x_1 + x_2 + x_3$, $3\bar{y} = y_1 + y_2 + y_3$.]

Question (2000 STEP III Q1)

Sketch on the same axes the two curves C_1 and C_2, given by

$$C_1 : \qquad xy = 1$$
$$C_2 : \qquad x^2 - y^2 = 2$$

The curves intersect at P and Q. Given that the coordinates of P are (a, b) (which you need not evaluate), write down the coordinates of Q in terms of a and b.

The tangent to C_1 through P meets the tangent to C_2 through Q at the point M, and the tangent to C_2 through P meets the tangent to C_1 through Q at N. Show that the coordinates of M are $(-b, a)$ and write down the coordinates of N.

Show that $PMQN$ is a square.

Question (2001 STEP I Q1)

The points A, B and C lie on the sides of a square of side 1 cm and no two points lie on the same side. Show that the length of at least one side of the triangle ABC must be less than or equal to $(\sqrt{6} - \sqrt{2})$ cm.

Question (2002 STEP I Q1)

Show that the equation of any circle passing through the points of intersection of the ellipse $(x + 2)^2 + 2y^2 = 18$ and the ellipse $9(x - 1)^2 + 16y^2 = 25$ can be written in the form

$$x^2 - 2ax + y^2 = 5 - 4a\ .$$

Question (2002 STEP I Q6)

A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three sides of the pyramid are of length b and its volume is V. Given that the formula for the volume of any pyramid is $\frac{1}{3} \times$ area of base $\times$ height, show that

$$V = \frac{1}{12}a^2(3b^2 - a^2)^{\frac{1}{2}} .$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Show that the new height, h, of the pyramid is given by

$$h^2 = \frac{a^2(3b^2 - a^2)}{4b^2 - a^2} .$$

Find, in terms of a and b, the angle between the equilateral triangle and the horizontal.

Question (2003 STEP II Q4)

The line $y = d$, where $d > 0$, intersects the circle $x^2 + y^2 = R^2$ at G and H. Show that the area of the minor segment GH is equal to

$$R^2 \arccos\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2} . \qquad (*)$$

In the following cases, the given line intersects the given circle. Determine how, in each case, the expression $(*)$ should be modified to give the area of the minor segment.

(i) Line: $y = c$; circle: $(x - a)^2 + (y - b)^2 = R^2$.

(ii) Line: $y = mx + c$; circle: $x^2 + y^2 = R^2$.

(iii) Line: $y = mx + c$; circle: $(x - a)^2 + (y - b)^2 = R^2$.

Question (2004 STEP I Q6)

The three points A, B and C have coordinates (p_1 , q_1), (p_2 , q_2) and (p_3 , q_3), respectively. Find the point of intersection of the line joining A to the midpoint of BC, and the line joining B to the midpoint of AC. Verify that this point lies on the line joining C to the midpoint of AB.

The point H has coordinates $(p_1 + p_2 + p_3 , q_1 + q_2 + q_3)$. Show that if the line AH intersects the line BC at right angles, then $p_2^2 + q_2^2 = p_3^2 + q_3^2$, and write down a similar result if the line BH intersects the line AC at right angles.

Deduce that if AH is perpendicular to BC and also BH is perpendicular to AC, then CH is perpendicular to AB.

Question (2005 STEP I Q6) **(i)** The point A has coordinates $(5\,,16)$ and the point B has coordinates $(-4\,,4)$. The variable point P has coordinates $(x\,,y)$ and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is

$$(x+7)^2 + y^2 = 100 \ .$$

(ii) The point C has coordinates $(a\,,0)$ and the point D has coordinates $(b\,,0)$, where $a \neq b$. The variable point Q moves on a path such that

$$QC = k \times QD \ ,$$

where $k > 1$. Given that the path of Q is the same as the path of P, show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51} \ .$$

Show further that $(a+7)(b+7) = 100$.

Question (2005 STEP II Q5)

The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1) \ ,$$

where $q = \dfrac{b+c}{a}$. Deduce that

$$R \leq \frac{1}{\pi(\pi-1)} \ .$$

Question (2013 STEP II Q4)

The line passing through the point $(a, 0)$ with gradient b intersects the circle of unit radius centred at the origin at P and Q, and M is the midpoint of the chord PQ. Find the coordinates of M in terms of a and b.

(i) Suppose b is fixed and positive. As a varies, M traces out a curve (the *locus* of M). Show that $x = -by$ on this curve. Given that a varies with $-1 \le a \le 1$, show that the locus is a line segment of length $2b/(1 + b^2)^{\frac{1}{2}}$. Give a sketch showing the locus and the unit circle.

(ii) Find the locus of M in the following cases, giving in each case its cartesian equation, describing it geometrically and sketching it in relation to the unit circle:

- a is fixed with $0 < a < 1$, and b varies with $-\infty < b < \infty$;
- $ab = 1$, and b varies with $0 < b \le 1$.

Question (2015 STEP II Q7)

A circle C is said to be *bisected* by a curve X if X meets C in exactly two points and these points are diametrically opposite each other on C.

(i) Let C be the circle of radius a in the x-y plane with centre at the origin.

Show, by giving its equation, that it is possible to find a circle of given radius r that bisects C provided $r > a$. Show that no circle of radius r bisects C if $r \le a$.

(ii) Let C_1 and C_2 be circles with centres at $(-d, 0)$ and $(d, 0)$ and radii a_1 and a_2, respectively, where $d > a_1$ and $d > a_2$. Let D be a circle of radius r that bisects both C_1 and C_2. Show that the x-coordinate of the centre of D is $\dfrac{a_2^2 - a_1^2}{4d}$.

Obtain an expression in terms of d, r, a_1 and a_2 for the y-coordinate of the centre of D, and deduce that r must satisfy

$$16r^2d^2 \ge \left(4d^2 + (a_2 - a_1)^2\right)\left(4d^2 + (a_2 + a_1)^2\right).$$

Question (2016 STEP I Q5)

(i) The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

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[tl](0.892440731042,3.12683570963)Q [tl](1.01215640631,3.12523930063)R

Show that

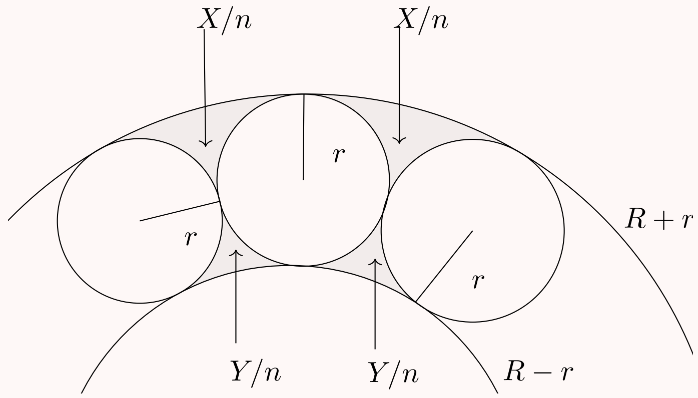
$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < a < c$, which satisfy $(**)$. Show that they also satisfy $(*)$.

Question (1987 STEP I Q2)



The region A between concentric circles of radii $R+r$, $R-r$ contains n circles of radius r. Each circle of radius r touches both of the larger circles as well as its two neighbours of radius r, as shown in the figure. Find the relationship which must hold between n, R and r. Show that Y, the total area of A outside the circle of radius r and adjacent to the circle of radius $R-r$, is given by

$$Y = nr\sqrt{R^2 - r^2} - \pi(R-r)^2 - n\pi r^2\left(\frac{1}{2} - \frac{1}{n}\right).$$

Find similar expressions for X, the total area of A outside the circles of radius r and adjacent to the circle of radius $R+r$, and for Z, the total area inside the circle of radius r. What value does $(X+Y)/Z$ approach when n becomes large?

Question (2025 STEP III Q3)

Let $f(x)$ be defined and positive for $x > 0$. Let a and b be real numbers with $0 < a < b$ and define the points $A = (a, f(a))$ and $B = (b, -f(b))$. Let $X = (m, 0)$ be the point of intersection of line AB with the x-axis.

(i) Find an expression for m in terms of a, b, $f(a)$ and $f(b)$.

(ii) Show that, if $f(x) = \sqrt{x}$, then $m = \sqrt{ab}$. Find, in terms of n, a function $f(x)$ such that $m = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$.

(iii) Let $g_1(x)$ and $g_2(x)$ be defined and positive for $x > 0$. Let $m = M_1$ when $f(x) = g_1(x)$ and let $m = M_2$ when $f(x) = g_2(x)$. Show that if $\frac{g_1(x)}{g_2(x)}$ is a decreasing function then $M_1 > M_2$. Hence show that

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}$$

(iv) Let p and c be chosen so that the curve $y = p(c-x)^3$ passes through both A and B. Show that

$$\frac{c-a}{b-c} = \left(\frac{f(a)}{f(b)}\right)^{1/3}$$

and hence determine c in terms of a, b, $f(a)$ and $f(b)$. Show that if f is a decreasing function, then $c < m$.