

**Question (1991 STEP III Q10)**

The equation

$$x^n - qx^{n-1} + r = 0,$$

where  $n \geq 5$  and  $q$  and  $r$  are real constants, has roots  $\alpha_1, \alpha_2, \dots, \alpha_n$ . The sum of the products of  $m$  distinct roots is denoted by  $\Sigma_m$  (so that, for example,  $\Sigma_3 = \sum \alpha_i \alpha_j \alpha_k$  where the sum runs over the values of  $i, j$  and  $k$  with  $n \geq i > j > k \geq 1$ ). The sum of  $m$ th powers of the roots is denoted by  $S_m$  (so that, for example,  $S_3 = \sum_{i=1}^n \alpha_i^3$ ).

Prove that  $S_p = q^p$  for  $1 \leq p \leq n-1$ .

*You may assume that for any  $n$ th degree equation and  $1 \leq p \leq n$*

$$S_p - S_{p-1}\Sigma_1 + S_{p-2}\Sigma_2 - \dots + (-1)^{p-1}S_1\Sigma_{p-1} + (-1)^p p \Sigma_p = 0.]$$

Find expressions for  $S_n, S_{n+1}$  and  $S_{n+2}$  in terms of  $q, r$  and  $n$ . Suggest an expression for  $S_{n+m}$ , where  $m < n$ , and prove its validity by induction.

**Question (1992 STEP II Q7)**

The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots  $a, b$  and  $c$ . Express  $p, q$  and  $r$  in terms of  $a, b$  and  $c$ .

(i) If  $p = 0$  and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

**Question (1996 STEP III Q7)** (i) If  $x + y + z = \alpha$ ,  $xy + yz + zx = \beta$  and  $xyz = \gamma$ , find numbers  $A, B$  and  $C$  such that

$$x^3 + y^3 + z^3 = A\alpha^3 + B\alpha\beta + C\gamma.$$

Solve the equations

$$\begin{aligned}x + y + z &= 1 \\x^2 + y^2 + z^2 &= 3 \\x^3 + y^3 + z^3 &= 4.\end{aligned}$$

(ii) The area of a triangle whose sides are  $a, b$  and  $c$  is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where  $s$  is the semi-perimeter  $\frac{1}{2}(a + b + c)$ . If  $a, b$  and  $c$  are the roots of the equation

$$x^3 - 16x^2 + 81x - 128 = 0,$$

find the area of the triangle.

**Question (1997 STEP III Q4)**

In this question, you may assume that if  $k_1, \dots, k_n$  are distinct positive real numbers, then

$$\frac{1}{n} \sum_{r=1}^n k_r > \left( \prod_{r=1}^n k_r \right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean. Suppose that  $a, b, c$  and  $d$  are positive real numbers such that the polynomial

$$f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots.

- (i) Show that  $pqr, qrs, rsp$  and  $spq$  are distinct, where  $p, q, r$  and  $s$  are the roots of the polynomial  $f$ .
- (ii) By considering the relationship between the coefficients of  $f$  and its roots, show that  $c > d$ .
- (iii) Explain why the polynomial  $f'(x)$  must have three distinct roots.
- (iv) By differentiating  $f$ , show that  $b > c$ .
- (v) Show that  $a > b$ .

**Question (2007 STEP III Q1)**

In this question, do not consider the special cases in which the denominators of any of your expressions are zero. Express  $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$  in terms of  $t_i$ , where  $t_1 = \tan \theta_1$ , etc. Given that  $\tan \theta_1, \tan \theta_2, \tan \theta_3$  and  $\tan \theta_4$  are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where  $a \neq 0$ ), find an expression in terms of  $a, b, c, d$  and  $e$  for  $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ . The four real numbers  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  lie in the range  $0 \leq \theta_i < 2\pi$  and satisfy the equation

$$p \cos 2\theta + \cos(\theta - \alpha) + p = 0,$$

where  $p$  and  $\alpha$  are independent of  $\theta$ . Show that  $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$  for some integer  $n$ .

**Question (2008 STEP III Q1)**

Find all values of  $a, b, x$  and  $y$  that satisfy the simultaneous equations

$$\begin{aligned} a + b &= 1 \\ ax + by &= \frac{1}{3} \\ ax^2 + by^2 &= \frac{1}{5} \\ ax^3 + by^3 &= \frac{1}{7}. \end{aligned}$$

[ **Hint:** you may wish to start by multiplying the second equation by  $x + y$ . ]

**Question (2009 STEP III Q5)**

The numbers  $x, y$  and  $z$  satisfy

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 2 \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Show that

$$yz + zx + xy = -\frac{1}{2}.$$

Show also that  $x^2y + x^2z + y^2z + y^2x + z^2x + z^2y = -1$ , and hence that

$$xyz = \frac{1}{6}.$$

Let  $S_n = x^n + y^n + z^n$ . Use the above results to find numbers  $a, b$  and  $c$  such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2},$$

holds for all  $n$ .

**Question (2014 STEP III Q1)**

Let  $a$ ,  $b$  and  $c$  be real numbers such that  $a + b + c = 0$  and let

$$(1 + ax)(1 + bx)(1 + cx) = 1 + qx^2 + rx^3$$

for all real  $x$ . Show that  $q = bc + ca + ab$  and  $r = abc$ .

- (i) Show that the coefficient of  $x^n$  in the series expansion (in ascending powers of  $x$ ) of  $\ln(1 + qx^2 + rx^3)$  is  $(-1)^{n+1}S_n$  where

$$S_n = \frac{a^n + b^n + c^n}{n}, \quad (n \geq 1).$$

- (ii) Find, in terms of  $q$  and  $r$ , the coefficients of  $x^2$ ,  $x^3$  and  $x^5$  in the series expansion (in ascending powers of  $x$ ) of  $\ln(1 + qx^2 + rx^3)$  and hence show that  $S_2S_3 = S_5$ .
- (iii) Show that  $S_2S_5 = S_7$ .
- (iv) Give a proof of, or find a counterexample to, the claim that  $S_2S_7 = S_9$ .

**Question (2015 STEP III Q6)** (i) Let  $w$  and  $z$  be complex numbers, and let  $u = w + z$  and  $v = w^2 + z^2$ . Prove that  $w$  and  $z$  are real if and only if  $u$  and  $v$  are real and  $u^2 \leq 2v$ .

- (ii) The complex numbers  $u$ ,  $w$  and  $z$  satisfy the equations

$$\begin{aligned} w + z - u &= 0 \\ w^2 + z^2 - u^2 &= -\frac{2}{3} \\ w^3 + z^3 - \lambda u &= -\lambda \end{aligned}$$

where  $\lambda$  is a positive real number. Show that for all values of  $\lambda$  except one (which you should find) there are three possible values of  $u$ , all real.

Are  $w$  and  $z$  necessarily real? Give a proof or counterexample.

**Question (2017 STEP I Q4)** (i) Let  $r$  be a real number with  $|r| < 1$  and let

$$S = \sum_{n=0}^{\infty} r^n.$$

You may assume without proof that  $S = \frac{1}{1-r}$ . Let  $p = 1 + r + r^2$ . Sketch the graph of the function  $1 + r + r^2$  and deduce that  $\frac{3}{4} \leq p < 3$ . Show that, if  $1 < p < 3$ , then the value of  $p$  determines  $r$ , and hence  $S$ , uniquely. Show also that, if  $\frac{3}{4} < p < 1$ , then there are two possible values of  $S$  and these values satisfy the equation  $(3-p)S^2 - 3S + 1 = 0$ .

(ii) Let  $r$  be a real number with  $|r| < 1$  and let

$$T = \sum_{n=1}^{\infty} nr^{n-1}.$$

You may assume without proof that  $T = \frac{1}{(1-r)^2}$ . Let  $q = 1 + 2r + 3r^2$ . Find the set of values of  $q$  that determine  $T$  uniquely. Find the set of values of  $q$  for which  $T$  has two possible values. Find also a quadratic equation, with coefficients depending on  $q$ , that is satisfied by these two values.

**Question (2017 STEP III Q3)**

Let  $\alpha, \beta, \gamma$  and  $\delta$  be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

You are given that, for any such equation,  $\alpha\beta + \gamma\delta$ ,  $\alpha\gamma + \beta\delta$  and  $\alpha\delta + \beta\gamma$  satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr - 4s)y + (4qs - p^2s - r^2) = 0.$$

Determine  $A$ . Now consider the quartic equation given by  $p = 0$ ,  $q = 3$ ,  $r = -6$  and  $s = 10$ .

- (i) Find the value of  $\alpha\beta + \gamma\delta$ , given that it is the largest root of the corresponding cubic equation.
- (ii) Hence, using the values of  $q$  and  $s$ , find the value of  $(\alpha + \beta)(\gamma + \delta)$  and the value of  $\alpha\beta$  given that  $\alpha\beta > \gamma\delta$ .
- (iii) Using these results, and the values of  $p$  and  $r$ , solve the quartic equation.

**Question (2018 STEP I Q7)** (i) In the cubic equation  $x^3 - 3pqx + pq(p + q) = 0$ , where  $p$  and  $q$  are distinct real numbers, use the substitution

$$x = \frac{pz + q}{z + 1}$$

to show that the equation reduces to  $az^3 + b = 0$ , where  $a$  and  $b$  are to be expressed in terms of  $p$  and  $q$ .

(ii) Show further that the equation  $x^3 - 3cx + d = 0$ , where  $c$  and  $d$  are non-zero real numbers, can be written in the form  $x^3 - 3pqx + pq(p + q) = 0$ , where  $p$  and  $q$  are distinct real numbers, provided  $d^2 > 4c^3$ .

(iii) Find the real root of the cubic equation  $x^3 + 6x - 2 = 0$ .

(iv) Find the roots of the equation  $x^3 - 3p^2x + 2p^3 = 0$ , and hence show how the equation  $x^3 - 3cx + d = 0$  can be solved in the case  $d^2 = 4c^3$ .

**Question (2018 STEP III Q1)** (i) The function  $f$  is given by

$$f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \quad (\beta \neq 0).$$

Find the stationary point of the curve  $y = f(\beta)$  and sketch the curve. Sketch also the curve  $y = g(\beta)$ , where

$$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \quad (\beta \neq 0).$$

(ii) Let  $u$  and  $v$  be the roots of the equation

$$x^2 + \alpha x + \beta = 0,$$

where  $\beta \neq 0$ . Obtain expressions in terms of  $\alpha$  and  $\beta$  for  $u + v + \frac{1}{uv}$  and  $\frac{1}{u} + \frac{1}{v} + uv$ .

(iii) Given that  $u + v + \frac{1}{uv} = -1$ , and that  $u$  and  $v$  are real, show that  $\frac{1}{u} + \frac{1}{v} + uv \leq -1$ .

(iv) Given instead that  $u + v + \frac{1}{uv} = 3$ , and that  $u$  and  $v$  are real, find the greatest value of  $\frac{1}{u} + \frac{1}{v} + uv$ .

**Question (2019 STEP III Q4)**

The  $n$ th degree polynomial  $P(x)$  is said to be *reflexive* if:

- (a)  $P(x)$  is of the form  $x^n - a_1x^{n-1} + a_2x^{n-2} - \dots + (-1)^na_n$  where  $n \geq 1$ ;
  - (b)  $a_1, a_2, \dots, a_n$  are real;
  - (c) the  $n$  (not necessarily distinct) roots of the equation  $P(x) = 0$  are  $a_1, a_2, \dots, a_n$ .
- (i) Find all reflexive polynomials of degree less than or equal to 3.
- (ii) For a reflexive polynomial with  $n > 3$ , show that

$$2a_2 = -a_2^2 - a_3^2 - \dots - a_n^2.$$

Deduce that, if all the coefficients of a reflexive polynomial of degree  $n$  are integers and  $a_n \neq 0$ , then  $n \leq 3$ .

- (iii) Determine all reflexive polynomials with integer coefficients.

**Question (2025 STEP III Q6)** (i) Let  $a, b$  and  $c$  be three non-zero complex numbers with the properties  $a + b + c = 0$  and  $a^2 + b^2 + c^2 = 0$ . Show that  $a, b$  and  $c$  cannot all be real. Show further that  $a, b$  and  $c$  all have the same modulus.

- (ii) Show that it is not possible to find three non-zero complex numbers  $a, b$  and  $c$  with the properties  $a + b + c = 0$  and  $a^3 + b^3 + c^3 = 0$ .
- (iii) Show that if any four non-zero complex numbers  $a, b, c$  and  $d$  have the properties  $a + b + c + d = 0$  and  $a^3 + b^3 + c^3 + d^3 = 0$ , then at least two of them must have the same modulus.
- (iv) Show, by taking  $c = 1, d = -2$  and  $e = 3$  that it is possible to find five real numbers  $a, b, c, d$  and  $e$  with distinct magnitudes and with the properties  $a + b + c + d + e = 0$  and  $a^3 + b^3 + c^3 + d^3 + e^3 = 0$ .