Question (1990 STEP I Q8)

Show that
$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right) = \frac{\sin\alpha}{4\sin\left(\frac{\alpha}{4}\right)},$$
where $\alpha \neq k\pi$, k is an integer.

Prove that, for such α,
$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right) = \frac{\sin\alpha}{2^n\sin\left(\frac{\alpha}{2^n}\right)},$$
where n is a positive integer.

Deduce that
$$\alpha = \frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\cdots},$$
and hence that
$$\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}.$$

$$\begin{aligned}
\sin\alpha &= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\
&= 4\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}\cos\frac{\alpha}{2} \\
\Rightarrow \qquad \frac{\sin\alpha}{4\sin\frac{\alpha}{4}} &= \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}
\end{aligned}$$

We proceed by induction on n. Clearly this is true for $n = 1$ (as we just established). Assume it is true for $n = k$. Then:

$$\begin{aligned}
\frac{\sin\alpha}{2^n\sin\frac{\alpha}{2^n}} &= \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right) \\
\Rightarrow \qquad \frac{\sin\alpha}{2^{n+1}\sin\frac{\alpha}{2^{n+1}}\cos\frac{\alpha}{2^{n+1}}} &= \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right) \\
\Rightarrow \qquad \frac{\sin\alpha}{2^{n+1}\sin\frac{\alpha}{2^{n+1}}} &= \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right)\cos\left(\frac{\alpha}{2^{n+1}}\right)
\end{aligned}$$

Therefore it is true for $n = k+1$ Therefore since it is true for $n = 1$ and if it is true for $n = k$ it is also true for $n = k+1$ by the principle of mathematical induction it is true for all $n \geq 1$

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right)} &= \lim_{n\to\infty} 2^n\sin\frac{\alpha}{2^n} \\
&= \lim_{n\to\infty}\alpha\frac{\sin\frac{\alpha}{2^n}}{\frac{\alpha}{2^n}} \\
&= \alpha\lim_{t\to 0}\frac{\sin t}{t}
\end{aligned}$$

$$= \alpha$$

When $\alpha = \frac{\pi}{2}$ notice that $\sin\alpha = 1$, $\cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}}$ and $2\cos^2\frac{\alpha}{2^{n+1}} - 1 = \cos\frac{\alpha}{2} \Rightarrow \cos\frac{\alpha}{2^{n+1}} = \sqrt{\frac{1}{2} + \cos\frac{\alpha}{2^n}}$ exactly the series we see.

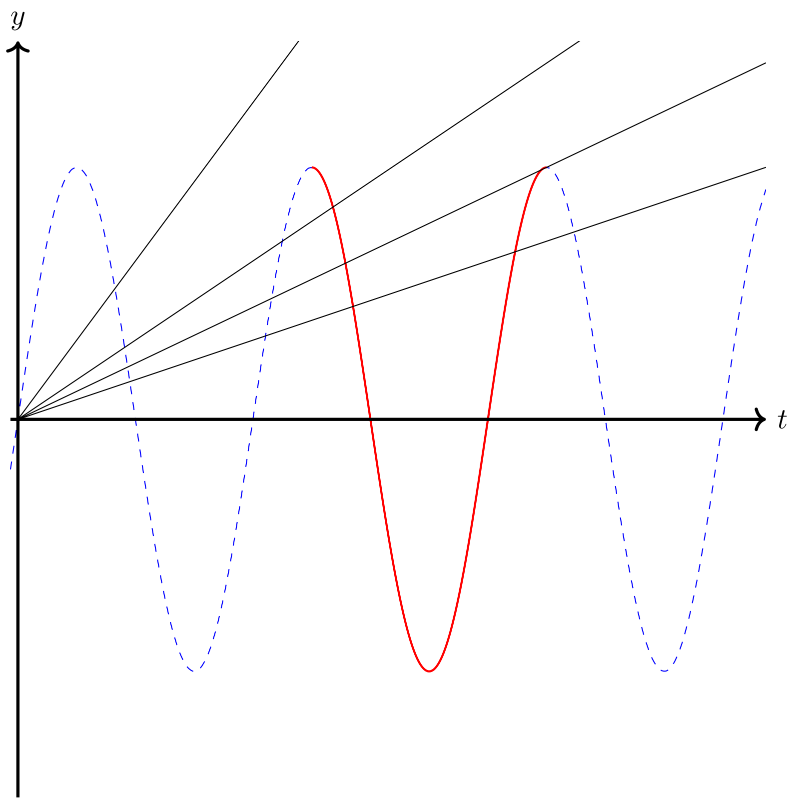
Question (1990 STEP III Q10)

By considering the graphs of $y = kx$ and $y = \sin x$, show that the equation $kx = \sin x$, where $k > 0$, may have $0, 1, 2$ or 3 roots in the interval $(4n+1)\frac{\pi}{2} < x < (4n+5)\frac{\pi}{2}$, where n is a positive integer.

For a certain given value of n, the equation has exactly one root in this interval. Show that k lies in an interval which may be written $\sin\delta < k < \dfrac{2}{(4n+1)\pi}$, where $0 < \delta < \frac{1}{2}\pi$ and

$$\cos\delta = \left((4n+5)\frac{\pi}{2} - \delta\right)\sin\delta.$$

Show that, if n is large, then $\delta \approx \dfrac{2}{(4n+5)\pi}$ and obtain a second, improved, approximation.



Clearly we can achieve 0, 1, and 2 intersections by never entering the range, entering too flat, or entering before hitting the second branch. To achieve 3 we can go at a flat enough slope that we hit somewhere near the top of the second branch, and since the gradient there will be ≈ 0, and our gradient is positive, we must intersect before that point as well, ie 3 intersections. Clearly we cannot intersect the second branch 3 times or the first branch twice, therefore there are at most 3 intersections.

To intersect the graph only once, we need to:

- be below $((4n+1)\frac{\pi}{2}, 1)$ and
- not touch the second gradient

The first condition means that $k(4n+1)\frac{\pi}{2} < 1 \Rightarrow k < \frac{2}{(4n+1)\pi}$. For the second condition, consider a point on the curve $\sin x$ whose tangent line goes through the origin, ie $\frac{y-\sin t}{x-t} = \cos t \Rightarrow y = (\cos t)x - t\cos t + \sin t$ ie $\sin t = t\cos t$. For this point t to be in the required interval, we need $(4n+5)\frac{\pi}{2} - t \in (0, \frac{\pi}{2})$, so let's call this value δ. Then our result is:

The gradient needs to be steeper than $\cos t = \cos\left((4n+5)\frac{\pi}{2} - \delta\right) = \sin\delta$ and $\cos\delta = \left((4n+5)\frac{\pi}{2} - \delta\right)\sin\delta$.

If n is large, then,

$$\begin{aligned} & 1 \approx \left((4n+5)\tfrac{\pi}{2} - \delta\right)\delta \\ \Rightarrow \quad & 1 \approx (4n+5)\tfrac{\pi}{2}\delta \\ \Rightarrow \quad & \delta \approx \frac{2}{(4n+5)\pi} \end{aligned}$$

.

To higher order:

$$\begin{aligned} & 1 - \frac{1}{2}\delta^2 \approx \left((4n+5)\tfrac{\pi}{2} - \delta\right)\delta \\ \Rightarrow \quad & 1 - \frac{1}{2}\delta^2 \approx (4n+5)\tfrac{\pi}{2}\delta - \delta^2 \\ \Rightarrow \quad & 0 \approx 1 - (4n+5)\tfrac{\pi}{2}\delta + \frac{1}{2}\delta^2 \\ \Rightarrow \quad & \delta \approx (4n+5)\tfrac{\pi}{2} - \sqrt{(4n+5)^2\frac{\pi^2}{4} - 2} \\ & = \frac{2}{(4n+5)\frac{\pi}{2} + \sqrt{(4n+5)^2\frac{\pi^2}{4} - 2}} \end{aligned}$$

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Question (1992 STEP I Q6)

Explain briefly, by means of a diagram, or otherwise, why

$$f(\theta + \delta\theta) \approx f(\theta) + f'(\theta)\delta\theta,$$

when $\delta\theta$ is small.

Two powerful telescopes are placed at points A and B which are a distance a apart. A very distant point C is such that AC makes an angle θ with AB and BC makes an angle $\theta + \phi$ with AB produced. (A sketch of the arrangement is given in the diagram.)

xunit=0.8cm,yunit=0.8cm,algebraic=true,dimen=middle,dotstyle=o,dotsize=3pt 0,linewidth=0.5pt,arrowsize=3pt 2,arrowinset=0.25 (-4.18,-0.94)(4.4,5.22) (-4,0)(4,0) (-2,0)(2,5) (2,5)(1,0) [tl](-2.3,-0.14)A [tl](1.08,-0.14)B [tl](-1.6,0.46)θ [tl](1.24,0.52)$\theta + \phi$ [tl](2.14,5.1)C

If the perpendicular distance h of C from AB is very large compared with a show that h is approximately $(a \sin^2 \theta)/\phi$ and find the approximate value of AC in terms of a, θ and ϕ.

It is easy to show (but you are not asked to show it) that errors in measuring ϕ are much more important than errors in measuring θ. If we make an error of $\delta\phi$ in measuring ϕ (but measure θ correctly) what is the approximate error in our estimate of AC and, roughly, in what proportion is it reduced by doubling the distance between A and B?

Question (1992 STEP II Q1)

Find the limit, as $n \to \infty$, of each of the following. You should explain your reasoning briefly.

(i) $\dfrac{n}{n+1}$, **(ii)** $\dfrac{5n+1}{n^2-3n+4}$, **(iii)** $\dfrac{\sin n}{n}$,

(iv) $\dfrac{\sin(1/n)}{(1/n)}$, **(v)** $(\arctan n)^{-1}$, **(vi)** $\dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}-\sqrt{n}}$.

(i)

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\ &\underbrace{=}_{\text{sum of limits}} \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} \\ &= 1\end{aligned}$$

(ii)

$$\lim_{n\to\infty} \frac{5n+1}{n^2-3n+4} = \lim_{n\to\infty} \frac{5/n + 1/n^2}{1 - 3/n + 4/n^2}$$

$$\underbrace{=}_{\text{ratio of limits}} \frac{\lim_{n\to\infty}(5/n + 1/n^2)}{\lim_{n\to\infty}(1 - 3/n + 4/n^2)}$$
$$= \frac{0}{1} = 0$$

(iii)

$$|\frac{\sin n}{n}| \leq \frac{1}{n} \qquad (n \geq 1)$$
$$\Rightarrow \qquad \lim_{n\to\infty} |\frac{\sin n}{n}| \leq \lim_{n\to\infty} \frac{1}{n}$$
$$= 0$$
$$\Rightarrow \qquad \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

(iv) First note that $\lim_{x\to 0} \dfrac{\sin x}{x} \to 1$, then $\frac{1}{n}$ is a sequence converging to zero, therefore $\frac{\sin 1/n}{1/n}$ also must tend to 1.

(v) Note that $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ and since n is a sequence tending to infinity we must have $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$

(vi)

$$\lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+2} - \sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{2}{\sqrt{n+2}+\sqrt{n}}}$$
$$= \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$
$$= \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt{1 + 2/n} + \sqrt{1}}{\sqrt{1 + 1/n} + \sqrt{1}}$$
$$= \frac{1}{2}$$

Question (1999 STEP I Q5)

For this question, you may use the following approximations, valid if θ is small: $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$. A satellite X is directly above the point Y on the Earth's surface and can just be seen (on the horizon) from another point Z on the Earth's surface. The radius of the Earth is R and the height of the satellite above the Earth is h.

(i) Find the distance d of Z from Y along the Earth's surface.

(ii) If the satellite is in low orbit (so that h is small compared with R), show that

$$d \approx k(Rh)^{1/2},$$

where k is to be found.

(iii) If the satellite is very distant from the Earth (so that R is small compared with h), show that

$$d \approx aR + b(R^2/h),$$

where a and b are to be found.

Question (2000 STEP II Q3)

The lengths of the sides BC, CA, AB of the triangle ABC are denoted by a, b, c, respectively. Given that

$$b = 8 + \epsilon_1,\ c = 3 + \epsilon_2,\ A = \tfrac{1}{3}\pi + \epsilon_3,$$

where ϵ_1, ϵ_2, and ϵ_3 are small, show that $a \approx 7+\eta$, where $\eta = \left(13\,\epsilon_1 - 2\,\epsilon_2 + 24\sqrt{3}\,\epsilon_3\right)/14$. Given now that

$$|\epsilon_1| \leq 2\times 10^{-3}, \quad |\epsilon_2| \leq 4\cdot 9\times 10^{-2}, \quad |\epsilon_3| \leq \sqrt{3}\times 10^{-3},$$

find the range of possible values of η.

The cosine rule states that:

$a^2 = b^2 + c^2 - 2bc\cos(A)$

Therefore

$$\begin{aligned}
a^2 &= (8+\epsilon_1)^2 + (3+\epsilon_2)^2 - 2(8+\epsilon_1)(3+\epsilon_2)\cos\left(\frac{\pi}{3}+\epsilon_3\right)\\
&\approx 64 + 16\epsilon_1 + 9 + 6\epsilon_2 - 2(24 + 3\epsilon_1 + 8\epsilon_2)\cos\left(\frac{\pi}{3}+\epsilon_3\right)\\
&= 73 + 16\epsilon_1 + 6\epsilon_2 - 2(24 + 3\epsilon_1 + 8\epsilon_2)\left(\cos\left(\frac{\pi}{3}\right)\cos\epsilon_3 - \sin\left(\frac{\pi}{3}\right)\sin\epsilon_3\right)\\
&\approx 73 + 16\epsilon_1 + 6\epsilon_2 - (24 + 3\epsilon_1 + 8\epsilon_2) + 24\sqrt{3}\epsilon_3\\
&= 49 + 13\epsilon_1 - 2\epsilon_2 + 24\sqrt{3}\epsilon_3\\
&= 7^2 + 2\cdot 7\cdot\frac{13\epsilon_1 - 2\epsilon_2 + 24\sqrt{3}\epsilon_3}{14}
\end{aligned}$$

$$\approx \left(7 + \frac{13\epsilon_1 - 2\epsilon_2 + 24\sqrt{3}\epsilon_3}{14}\right)^2$$

In this approximation, we are ignoring all terms of order 2, and using the approximations $\cos\varepsilon \approx 1, \sin\varepsilon \approx \varepsilon$

Therefore $a \approx 7 + \frac{13\epsilon_1 - 2\epsilon_2 + 24\sqrt{3}\epsilon_3}{14}$.

η is maximised if ϵ_1, ϵ_3 are and ϵ_2 is minimized, ie:

$$\begin{aligned}\eta &\leq \frac{13 \cdot 2 \cdot 10^{-3} - 2 \cdot 4.9 \cdot 10^{-2} + 24\sqrt{3} \cdot \sqrt{3} \cdot 10^{-3}}{14} \\ &= 10^{-3} \cdot \frac{26 - 98 + 74}{14} \\ &= 10^{-3} \cdot \frac{1}{7}\end{aligned}$$

Similarly, it is maximised when signs are reversed, ie:

$|\eta| \leq 10^{-3} \cdot \frac{1}{7}$

Question (2009 STEP II Q2)

The curve C has equation

$$y = a^{\sin(\pi e^x)},$$

where $a > 1$.

(i) Find the coordinates of the stationary points on C.

(ii) Use the approximations $e^t \approx 1 + t$ and $\sin t \approx t$ (both valid for small values of t) to show that

$$y \approx 1 - \pi x \ln a$$

for small values of x.

(iii) Sketch C.

(iv) By approximating C by means of straight lines joining consecutive stationary points, show that the area between C and the x-axis between the kth and $(k+1)$th maxima is approximately

$$\left(\frac{a^2+1}{2a}\right) \ln\left(1 + (k - \tfrac{3}{4})^{-1}\right).$$