Question (1991 STEP III Q10)

The equation

$$x^n - qx^{n-1} + r = 0,$$

where $n \geqslant 5$ and q and r are real constants, has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. The sum of the products of m distinct roots is denoted by Σ_m (so that, for example, $\Sigma_3 = \sum \alpha_i\alpha_j\alpha_k$ where the sum runs over the values of i, j and k with $n \geqslant i > j > k \geqslant 1$). The sum of mth powers of the roots is denoted by S_m (so that, for example, $S_3 = \sum_{i=1}^{n} \alpha_i^3$).

Prove that $S_p = q^p$ for $1 \leqslant p \leqslant n-1$.

You may assume that for any nth degree equation and $1 \leqslant p \leqslant n$

$$S_p - S_{p-1}\Sigma_1 + S_{p-2}\Sigma_2 - \cdots + (-1)^{p-1}S_1\Sigma_{p-1} + (-1)^p p\Sigma_p = 0.]$$

Find expressions for S_n, S_{n+1} and S_{n+2} in terms of q, r and n. Suggest an expression for S_{n+m}, where $m < n$, and prove its validity by induction.

Claim: $S_p = q^p$ for $1 \leq p \leq n-1$

Proof: When $p = 1$, $S_p = \Sigma_1 = q$ as expected.

Note that $\Sigma_i = 0$ for $i = 2, \cdots, n-1$.

Using $S_p = S_{p-1}\Sigma_1 - S_{p-2}\Sigma_2 + \cdots + (-1)^{p-1+1}S_1\Sigma_{p-1} + (-1)^{p+1}p\Sigma_p$, we can see that $S_p = qS_{p-q}$ when $1 \leq p \leq n-1$, ie $S_p = q^p$.

Note that

$$\begin{aligned} S_n &= \sum \alpha_i^n \\ &= q\sum \alpha_i^{n-1} - \sum r \\ &= qS_{n-1} - nr \\ &= q^n - nr \end{aligned}$$

$$\begin{aligned} S_{n+1} &= \sum \alpha_i^{n+1} \\ &= q\sum \alpha_i^n - r\sum \alpha_i \\ &= q^{n+1} - rq \end{aligned}$$

$$\begin{aligned} S_{n+2} &= \sum \alpha_i^{n+2} \\ &= q\sum \alpha_i^{n+1} - r\sum \alpha_i^2 \\ &= q^{n+2} - rq^2 \end{aligned}$$

Claim: $S_{n+m} = q^{n+m} - rq^m$

Proof: The obvious

Question (1992 STEP II Q7)

The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a, b and c. Express p, q and r in terms of a, b and c.

(i) If $p = 0$ and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

$p = a + b + c, q = ab + bc + ca, r = abc$

(i) Suppose two roots are equal to each other, this means that one of the roots is also a root of the derivative. ie

$$\begin{aligned} 0 &= x^3 + qx - r \\ 0 &= 3x^2 + q \end{aligned}$$

have a common root, but this root must satisfy $x^2 = -\frac{q}{3}$. Then

$$\begin{aligned} & & 0 &= x^3 + qx - r \\ & & &= x^3 - 3x^3 - r \\ & & &= -2x^3 - r \\ \Rightarrow & & r^2 &= 4x^6 \\ & & &= 4\left(-\frac{q}{3}\right)^3 \\ \Rightarrow & & 0 &= 27r^2 + 4q^3 \end{aligned}$$

(ii) Consider $x = z + \frac{p}{3}$, then the equation is:

$$\begin{aligned} x^3 - px^2 + qx - r &= (z + \tfrac{p}{3})^3 - p(z + \tfrac{p}{3})^2 + q(z + \tfrac{p}{3}) - r \\ &= z^3 + pz^2 + \frac{p^2}{3}z + \frac{p^3}{27} - \\ &\quad - pz^2 - \frac{2p^2}{3}z - \frac{p^3}{9} + \\ &\quad qz + \frac{pq}{3} - r \\ &= z^3 + \left(\frac{p^2}{3} - \frac{2p^2}{3} + q\right)z + \left(\frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} - r\right) \\ &= z^3 + \left(-\frac{p^2}{3} + q\right)z + \left(-\frac{2p^3}{27} + \frac{pq}{3} - r\right) \end{aligned}$$

Since this equation must also have repeated roots we must have:

$$4\left(-\frac{p^2}{3}+q\right)^3+27\left(-\frac{2p^3}{27}+\frac{pq}{3}-r\right)^2=0$$

which is exactly our desired result

Question (1996 STEP III Q7) **(i)** If $x+y+z=\alpha$, $xy+yz+zx=\beta$ and $xyz=\gamma$, find numbers A, B and C such that

$$x^3+y^3+z^3=A\alpha^3+B\alpha\beta+C\gamma.$$

Solve the equations

$$\begin{aligned}x+y+z&=1\\x^2+y^2+z^2&=3\\x^3+y^3+z^3&=4.\end{aligned}$$

(ii) The area of a triangle whose sides are a, b and c is given by the formula

$$\text{area}=\sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter $\frac{1}{2}(a+b+c)$. If a, b and c are the roots of the equation

$$x^3-16x^2+81x-128=0,$$

find the area of the triangle.

(i)

$$\begin{aligned}(x+y+z)^3&=x^3+y^3+z^3+\\&\quad 3xy^2+3xz^2+3yx^2+\cdots+3zy^2\\&\quad +6xyz\\(x+y+z)(xy+yz+zx)&=x^2y+x^2z+\cdots+z^2x+3xyz\\x^3+y^3+z^3&=(x+y+z)^3-3(xy^2+\cdots+zy^2)-6xyz\\&=\alpha^3-3(\alpha\beta-3\gamma)-6\gamma\\&=\alpha^3-3\alpha\beta+3\gamma\end{aligned}$$

Since $4=1^3-3\cdot1\cdot(-1)+3\gamma\Rightarrow\gamma=0$, therefore one of $x, y, z=0$. WLOG $x=0$, so

$y+z=1, y^2+z^2=3\Rightarrow y^2+(1-y)^2=3\Rightarrow y^2-y-1=0\Rightarrow y=\frac{1\pm\sqrt{5}}{2}$, so we have

$(x,y,z)=(0,\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2})$ and permutations.

(ii)

$$A^2 = s(s-a)(s-b)(s-c)$$

Notice the second part is the same as plugging $s = 16/2 = 8$ into our polynomial

Therefore

$$\begin{aligned} A^2 &= 8 \cdot (8^3 - 16 \cdot 8^2 + 81 \cdot 8 - 128) \\ &= 8 \cdot 8(8^2 - 16 \cdot 8 + 81 - 16) \\ &= 64(-64 + 81 - 16) \\ &= 64 \end{aligned}$$

Therefore $A = 8$

Question (1997 STEP III Q4)

In this question, you may assume that if $k_1, \ldots, k_n$ are distinct positive real numbers, then

$$\frac{1}{n}\sum_{r=1}^{n} k_r > \left(\prod_{r=1}^{n} k_r\right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean. Suppose that a, b, c and d are positive real numbers such that the polynomial

$$\mathrm{f}(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots.

(i) Show that pqr, qrs, rsp and spq are distinct, where p, q, r and s are the roots of the polynomial f.

(ii) By considering the relationship between the coefficients of f and its roots, show that $c > d$.

(iii) Explain why the polynomial $\mathrm{f}'(x)$ must have three distinct roots.

(iv) By differentiating f, show that $b > c$.

(v) Show that $a > b$.

(i) Suppose $pqr = qrs$, since the roots are positive, we can divide by qr to obtain $p = s$ (a contradiction. Therefore all those terms are distinct.

(ii) $4c^3 = pqr + qrs + rsp + spq$, $d^4 = pqrs$.

Applying AM-GM, we obtain:

$$c^3 = \frac{pqr + qrs + rsp + spq}{4} > \sqrt[4]{p^3q^3r^3s^3} = d^3$$
$$\Rightarrow \qquad c > d$$

(iii) There must be a turning point between each root (since there are no repeated roots).

(iv) $f'(x) = 4x^3 - 12ax^2 + 12b^2 - 4c^3 = 4(x^3 - 3ax^2 + 3b^2 - c^3)$. Letting the roots of this polynomial be α, β, γ and again applying AM-GM, we must have:

$$b^2 = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{3} > \sqrt[3]{\alpha^2\beta^2\gamma^2} = c^2$$
$$\Rightarrow \qquad b > c$$

(v) Again, since there are turning points between the roots of $f'(x)$ we must have distinct roots for $f''(x)$, ie:

$f''(x) = 3x^2 - 6ax + 6b^2 = 3(x^2 - 2ax + b^2)$ has distinct real roots. But for this to occur we must have that $(2a)^2 - 4b^2 = 4(a^2 - b^2) > 0$, ie $a > b$

Question (2007 STEP III Q1)

In this question, do not consider the special cases in which the denominators of any of your expressions are zero. Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of t_i, where $t_1 = \tan\theta_1$, etc. Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of a, b, c, d and e for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$. The four real numbers θ_1, θ_2, θ_3 and θ_4 lie in the range $0 \leq \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0\,,$$

where p and α are independent of θ. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.

$$\begin{aligned}
\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) &= \frac{\tan(\theta_1 + \theta_2) + \tan(\theta_3 + \theta_4)}{1 - \tan(\theta_1 + \theta_2)\tan(\theta_3 + \theta_4)} \\
&= \frac{\frac{t_1+t_2}{1-t_1t_2} + \frac{t_3+t_4}{1-t_3t_4}}{1 - \frac{t_1+t_2}{1-t_1t_2}\frac{t_3+t_4}{1-t_3t_4}} \\
&= \frac{(t_1 + t_2)(1 - t_3t_4) + (t_3 + t_4)(1 - t_1t_2)}{(1 - t_1t_2)(1 - t_3t_4) - (t_1 + t_2)(t_3 + t_4)} \\
&= \frac{t_1 + t_2 + t_3 + t_4 - (t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)}{1 - t_1t_2 - t_1t_3 - t_1t_4 - t_2t_3 - t_2t_4 - t_3t_4}
\end{aligned}$$

If t_1, t_2, t_3, t_4 are the roots of $at^4 + bt^3 + ct^2 + dt + e = 0$, then $t_1 + t_2 + t_3 + t_4 = -\frac{b}{a}, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{c}{a}, t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 = -\frac{d}{a}$, therefore the expression is:

$$\begin{aligned}\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) &= \frac{-\frac{b}{a} + \frac{d}{a}}{1 - \frac{c}{a}} \\ &= \frac{d - b}{a - c}\end{aligned}$$

$$\begin{aligned}
0 &= p\cos 2\theta + \cos(\theta - \alpha) + p \\
&= p(2\cos^2\theta - 1) + \cos\theta\cos\alpha - \sin\theta\sin\alpha + p \\
&= 2p\cos^2\theta + \cos\theta\cos\alpha - \sin\theta\sin\alpha \\
\Rightarrow \quad 0 &= 2p\cos\theta + \cos\alpha - \tan\theta\sin\alpha \\
\Rightarrow \quad -2p\cos\theta &= \cos\alpha - \tan\theta\sin\alpha \\
\Rightarrow \quad 4p^2\cos^2\theta &= \cos^2\alpha - 2\sin\alpha\cos\alpha\tan\theta + \sin^2\alpha\tan^2\theta \\
4p^2\frac{1}{1+\tan^2\theta} &= \cos^2\alpha - \sin 2\alpha\tan\theta + \sin^2\alpha\tan^2\theta \\
\Rightarrow \quad 4p^2 &= \cos^2\alpha - \sin 2\alpha t + t^2 - \sin 2\alpha t^3 + \sin^2\alpha t^4 \\
\Rightarrow \quad \tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) &= \frac{0}{\sin^2\alpha - 1} \\
&= 0 \\
\Rightarrow \quad \theta_1 + \theta_2 + \theta_3 + \theta_4 &= n\pi
\end{aligned}$$

Question (2008 STEP III Q1)

Find all values of a, b, x and y that satisfy the simultaneous equations

$$\begin{aligned}
a + b &= 1 \\
ax + by &= \tfrac{1}{3} \\
ax^2 + by^2 &= \tfrac{1}{5} \\
ax^3 + by^3 &= \tfrac{1}{7}\,.
\end{aligned}$$

[**H**int: you may wish to start by multiplying the second equation by $x + y$.]

This is a second order recurrence relation, so we need to find m and n such that;

$$\begin{aligned}
\frac{1}{5} &= m\frac{1}{3} + n \\
\frac{1}{7} &= m\frac{1}{5} + n\frac{1}{3} \\
\Rightarrow \quad m, n &= \frac{6}{7}, -\frac{3}{35}
\end{aligned}$$

So we now need to solve the characteristic equation:
$\lambda^2 - \frac{6}{7}\lambda + \frac{3}{35} = 0$
So $x, y = \frac{15 \pm 2\sqrt{30}}{35}$.

We need,

$$\begin{aligned} & & 1 &= a+b \\ & & \frac{1}{3} &= a\frac{15+2\sqrt{30}}{35} + b\frac{15-2\sqrt{30}}{35} \\ & & \frac{1}{3} &= \frac{15}{35} + \frac{2\sqrt{30}}{35}(a-b) \\ \Rightarrow & & -\frac{\sqrt{30}}{18} &= a-b \\ \Rightarrow & & a &= \frac{18-\sqrt{30}}{36} \\ & & b &= \frac{18+\sqrt{30}}{38} \end{aligned}$$

So our two answers are:

$$(a,b,x,y) = \left(\frac{18\pm\sqrt{30}}{36}, \frac{18\mp\sqrt{30}}{36}, \frac{15\pm 2\sqrt{30}}{35}, \frac{15\mp 2\sqrt{30}}{35},\right)$$

Question (2009 STEP III Q5)

The numbers x, y and z satisfy

$$\begin{aligned} x+y+z &= 1 \\ x^2+y^2+z^2 &= 2 \\ x^3+y^3+z^3 &= 3\,. \end{aligned}$$

Show that

$$yz+zx+xy = -\frac{1}{2}\,.$$

Show also that $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y = -1$, and hence that

$$xyz = \frac{1}{6}\,.$$

Let $S_n = x^n+y^n+z^n$. Use the above results to find numbers a, b and c such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2}\,,$$

holds for all n.

$$\begin{aligned} & & (x+y+z)^2 &= x^2+y^2+z^2+2(xy+yz+zx) \\ \Rightarrow & & 1^2 &= 2+2(xy+yz+zx) \\ \Rightarrow & & xy+yz+zx &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} 1\cdot 2 &= (x+y+z)(x^2+y^2+z^2) \\ &= x^3+y^3+z^3+x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \end{aligned}$$

$$= 3 + x^2y + x^2z + y^2z + y^2x + z^2x + z^2y$$
$$\Rightarrow \quad -1 = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y$$

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + \cdots + 3zx^2 + 3zy^2 + 6xyz$$
$$\Rightarrow \quad 1 = 3 + 3(-1) + 6xyz$$
$$\Rightarrow \quad xyz = \frac{1}{6}$$

Since we have $f(t) = (t-x)(t-y)(t-z) = t^3 - t^2 - \frac{1}{2}t - \frac{1}{6}$ is zero for x, y, z we can notice that:
$t^{n+1} = t^n + \frac{1}{2}t^{n-1} + \frac{1}{6}t^{n-2}$ is also true for x, y, z (by multiplying by t^{n-2}.
Therefore:
$S_{n+1} = S_n + \frac{1}{2}S_{n-1} + \frac{1}{6}S_{n-2}$

Question (2014 STEP III Q1)

Let a, b and c be real numbers such that $a + b + c = 0$ and let

$$(1+ax)(1+bx)(1+cx) = 1 + qx^2 + rx^3$$

for all real x. Show that $q = bc + ca + ab$ and $r = abc$.

(i) Show that the coefficient of x^n in the series expansion (in ascending powers of x) of $\ln(1 + qx^2 + rx^3)$ is $(-1)^{n+1}S_n$ where

$$S_n = \frac{a^n + b^n + c^n}{n}, \qquad (n \geq 1).$$

(ii) Find, in terms of q and r, the coefficients of x^2, x^3 and x^5 in the series expansion (in ascending powers of x) of $\ln(1 + qx^2 + rx^3)$ and hence show that $S_2S_3 = S_5$.

(iii) Show that $S_2S_5 = S_7$.

(iv) Give a proof of, or find a counterexample to, the claim that $S_2S_7 = S_9$.

$$(1+ax)(1+bx)(1+cx) = (1 + (a+b)x + abx^2)(1+cx)$$
$$= 1 + (a+b+c)x + (ab+bc+ca)x^2 + abcx^3$$

Therefore by comparing coefficients, $q = bc + ca + ab$ and $r = abc$ as required.

(i)

$$\ln(1 + qx^2 + rx^3) = \ln(1+ax) + \ln(1+bx) + \ln(1+cx)$$
$$= -\sum_{n=1}^{\infty} \frac{(-ax)^n}{n} - \sum_{n=1}^{\infty} \frac{(-bx)^n}{n} - \sum_{n=1}^{\infty} \frac{(-cx)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a^n + b^n + c^n)}{n} x^n$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} S_n x^n$$

(ii)

$$\ln(1 + qx^2 + rx^3) = (qx^2 + rx^3) - \frac{(qx^2 + rx^3)^2}{2} + O(x^6)$$
$$= qx^2 + rx^3 - \frac{1}{2}q^2x^4 - qrx^5 + O(x^6)$$

Comparing coefficients we see that $S_2 = -q$ and $S_3 = r$, we also must have $S_5 = -qr = S_2S_3$ as required.

(iii)

$$\ln(1 + qx^2 + rx^3) = (qx^2 + rx^3) - \frac{(qx^2 + rx^3)^2}{2} + \frac{(qx^2 + rx^3)^3}{3} + O(x^8)$$
$$= qx^2 + rx^3 - \frac{1}{2}q^2x^4 - qrx^5 + \frac{1}{2}rx^6 + \frac{1}{3}q^3x^6 + q^2rx^7 + O(x^8)$$
$$= qx^2 + rx^3 - \frac{1}{2}q^2x^4 - qrx^5 + \left(\frac{1}{2}r + \frac{1}{3}q^3\right)x^6 + q^2rx^7$$

Comparing coefficients we see that $S_2 = -q$ and $S_5 = -qr$, we also must have $S_7 = q^2r = S_2S_5$ as required.

(iv) Let $a = b = 1, c = -2$, then $S_2 = \frac{1^2+1^2+(-2)^2}{2} = 3, S_7 = \frac{1^2+1^2+(-2)^7}{7} = -18, S_9 = \frac{1^1+1^2+(-2)^9}{9} = \frac{2-512}{9} \neq 3 \cdot (-18)$

Question (2015 STEP III Q6) **(i)** Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leq 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$w + z - u = 0$$
$$w^2 + z^2 - u^2 = -\tfrac{2}{3}$$
$$w^3 + z^3 - \lambda u = -\lambda$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u, all real.

Are w and z necessarily real? Give a proof or counterexample.

(i) Notice that $u^2 = v+2wz$, so w, z are roots of the quadratic $t^2 - ut + \frac{u^2-v}{2}$. Therefore they are both real if $u^2 \geq 2(u^2 - v) \Rightarrow 2v \geq u^2$.

(ii)

$$\begin{aligned} w + z &= u \\ w^2 + z^2 &= u^2 - \tfrac{2}{3} \\ w^3 + z^3 &= \lambda(u-1) \end{aligned}$$

$$wz = \frac{u^2 - (u^2 - \frac{2}{3})}{2} = \tfrac{1}{3}$$

$$\begin{aligned} (w+z)(w^2+z^2) &= w^3 + z^3 + wz(w+z) \\ u(u^2 - \tfrac{2}{3}) &= \lambda(u-1) + \frac{1}{3}u \\ \Rightarrow \quad u^3 - u &= \lambda(u-1) \\ \Rightarrow \quad 0 &= (u-1)(u(u+1) - \lambda) \\ \Rightarrow \quad 0 &= (u-1)(u^2 + u - \lambda) \end{aligned}$$

Therefore there will be at most 3 values for u, unless 1 is a root of $u^2 + u - \lambda$, ie $\lambda = 2$.

Suppose $u = 1$, then we have:

$w + z = 1, wz = 1/3 \Rightarrow w, z = \frac{-1 \pm \sqrt{-1/3}}{2}$ which are clearly complex.

Question (2017 STEP I Q4) **(i)** Let r be a real number with $|r| < 1$ and let

$$S = \sum_{n=0}^{\infty} r^n \, .$$

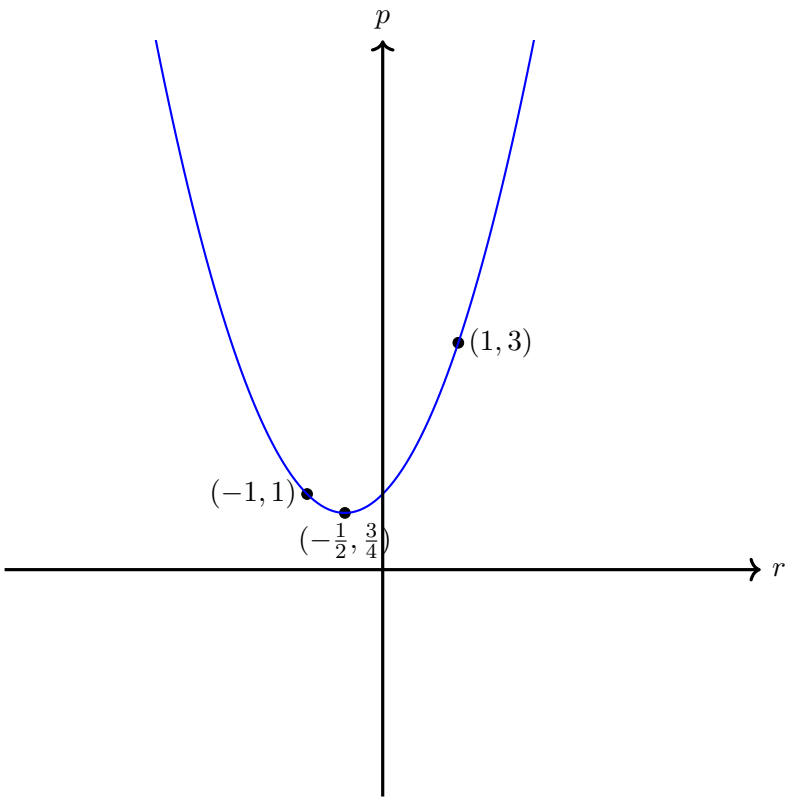
You may assume without proof that $S = \dfrac{1}{1-r}$. Let $p = 1 + r + r^2$. Sketch the graph of the function $1 + r + r^2$ and deduce that $\frac{3}{4} \leq p < 3$. Show that, if $1 < p < 3$, then the value of p determines r, and hence S, uniquely. Show also that, if $\frac{3}{4} < p < 1$, then there are two possible values of S and these values satisfy the equation $(3-p)S^2 - 3S + 1 = 0$.

(ii) Let r be a real number with $|r| < 1$ and let

$$T = \sum_{n=1}^{\infty} nr^{n-1} \, .$$

You may assume without proof that $T = \dfrac{1}{(1-r)^2}$. Let $q = 1 + 2r + 3r^2$. Find the set of values of q that determine T uniquely. Find the set of values of q for which T has two possible values. Find also a quadratic equation, with coefficients depending on q, that is satisfied by these two values.

(i)



Notice that $1 + r + r^2$ ranges from $\frac{3}{4}$ to 3 over $(-1, 1)$ therefore $\frac{3}{4} \leq p < 3$ attaining

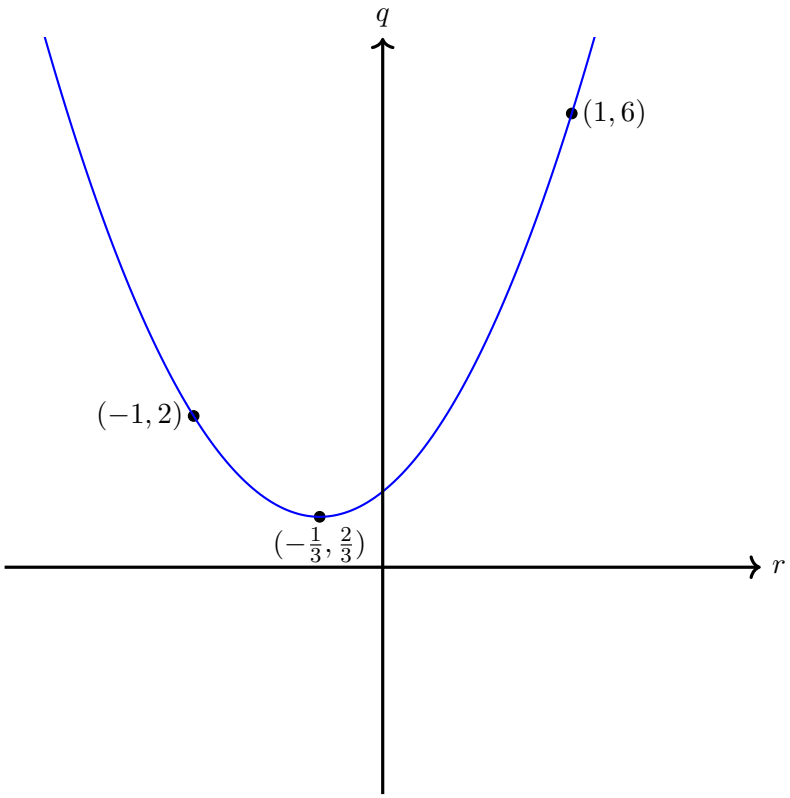
its minimum but not its maximum.

If $p > 1$ we know we must be on the right branch and hence we can determine r and S uniquely.

If $\frac{3}{4} < p < 1$ then we must have two possible values for r, satisfying $r_i^2 + r + 1 = p$ and so our two possible values for $S_i = \frac{1}{1-r_i}$ or $r_i = 1 - \frac{1}{S_i}$ and so

$$\begin{aligned} & p = \left(1 - \frac{1}{S}\right)^2 + 1 - \frac{1}{S} + 1 \\ \Rightarrow \quad & pS^2 = (S-1)^2 + S^2 - S + S^2 \\ \Rightarrow \quad & 0 = (3-p)S^2 - 3S + 1 \end{aligned}$$

(ii)



So $\frac{2}{3} \leq q < 6$ and r and hence T is uniquely determined if $2 \leq q < 6$. if $\frac{2}{3} < q < 2$ then there are two possible values, and they must satisfy:

$r = 1 - 1/\sqrt{T}$ so

$$\begin{aligned} q &= 1 + 2\left(1 - \frac{1}{\sqrt{T}}\right) + 3\left(1 - \frac{1}{\sqrt{T}}\right)^2 \\ &= 6 - \frac{2}{\sqrt{T}} - \frac{6}{\sqrt{T}} + \frac{3}{T} \\ &= 6 - \frac{8}{\sqrt{T}} + \frac{3}{T} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & (q-6)T - 3 = -8\sqrt{T} \\
\Rightarrow \quad & (q-6)^2T^2 - 6(q-6)T + 9 = 64T \\
\Rightarrow \quad & 0 = (q-6)^2T^2 - (6q+28)T + 9
\end{aligned}$$

Question (2017 STEP III Q3)

Let α, β, γ and δ be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0\,.$$

You are given that, for any such equation, $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr - 4s)y + (4qs - p^2s - r^2) = 0\,.$$

Determine A. Now consider the quartic equation given by $p = 0$, $q = 3$, $r = -6$ and $s = 10$.

(i) Find the value of $\alpha\beta + \gamma\delta$, given that it is the largest root of the corresponding cubic equation.

(ii) Hence, using the values of q and s, find the value of $(\alpha + \beta)(\gamma + \delta)$ and the value of $\alpha\beta$ given that $\alpha\beta > \gamma\delta$.

(iii) Using these results, and the values of p and r, solve the quartic equation.

$$\begin{aligned}
A &= -(\alpha\beta + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma) \\
&= -q
\end{aligned}$$

(i) The corresponding cubic equation is:

$$\begin{aligned}
0 &= y^3 - 3y^2 - 40y + (120 - 36) \\
&= y^3 - 3y^2 - 40y + 84 \\
&= (y-7)(y-2)(y+6)
\end{aligned}$$

Therefore $\alpha\beta + \gamma\delta = 7$

(ii)

$$\begin{aligned}
(\alpha + \beta)(\gamma + \delta) &= \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta \\
&= 3 - (\alpha\beta + \gamma\delta) \\
&= 3 - 7 = -4
\end{aligned}$$

Let $\alpha\beta$ and $\gamma\delta$ be the roots of a quadratic; then the quadratic will be $t^2 - 7t + 10 = 0 \Rightarrow t = 2, 5$ so $\alpha\beta = 5$

(iii) $\alpha\beta = 5, \gamma\delta = 2$

Consider the quadratic with roots $\alpha+\beta$ and $\gamma+\delta$, then

$t^2 - 4 = 0 \Rightarrow t = \pm 2$.

Suppose $\alpha+\beta = 2, \gamma+\delta = -2$ then

$\alpha, \beta = 1 \pm 2i, \gamma, \delta = -1 \pm i$

$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 5\gamma + 2\beta + 2\alpha + 5\delta = -6 \neq 6$

Suppose $\alpha+\beta = -2, \gamma+\delta = 2$ then

$\alpha, \beta = -1 \pm 2i, \gamma, \delta = 1 \pm i$

$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 5\gamma + 2\beta + 2\alpha + 5\delta = 6$, therefore these are there roots. (In some order):

$1 \pm i, -1 \pm 2i$

Question (2018 STEP I Q7) **(i)** In the cubic equation $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, use the substitution

$$x = \frac{pz+q}{z+1}$$

to show that the equation reduces to $az^3 + b = 0$, where a and b are to be expressed in terms of p and q.

(ii) Show further that the equation $x^3 - 3cx + d = 0$, where c and d are non-zero real numbers, can be written in the form $x^3 - 3pqx + pq(p+q) = 0$, where p and q are distinct real numbers, provided $d^2 > 4c^3$.

(iii) Find the real root of the cubic equation $x^3 + 6x - 2 = 0$.

(iv) Find the roots of the equation $x^3 - 3p^2x + 2p^3 = 0$, and hence show how the equation $x^3 - 3cx + d = 0$ can be solved in the case $d^2 = 4c^3$.

(i) Let $x = \frac{pz+q}{z+1}$ then

$$\begin{aligned}
0 &= x^3 - 3pqx + pq(p+q) \\
&= \left(\frac{pz+q}{z+1}\right)^3 - 3pq\left(\frac{pz+q}{z+1}\right) + pq(p+q) \\
&= \frac{(pz+q)^3 - 3pq(pz+q)(z+1)^2 + pq(p+q)(z+1)^3}{(z+1)^3} \\
&= \frac{1}{(z+1)^3}\Big((p^3 + pq(p+q) - 3p^2q)z^3 + (3p^2q - 6p^2q + 3pq^2 + 3p^2q + 3pq^2)z^2 + \\
&\qquad + (3pq^2 - 3p^2q - 6pq^2 + 3p^2q + 3qp^2)z + (q^3 - 3pq^2 + p^2q + pq^2)\Big)
\end{aligned}$$

$$\begin{aligned} &= \frac{(p^3+pq^2-2p^2q)z^3+(q^3+p^2q-2pq^2)}{(z+1)^3} \\ \Rightarrow \quad 0 &= (p^3+pq^2-2p^2q)z^3+(q^3+p^2q-2pq^2) \\ &= p(p-q)^2z^3+q(p-q)^2 \\ \Rightarrow \quad 0 &= pz^3+q \end{aligned}$$

(ii) We would like to find $pq=c$ and $pq(p+q)=d$, so p and q are roots of the quadratic $x^2-\frac{d}{c}x+c=0$, which has distinct real roots if $\Delta=\frac{d^2}{c^2}-4c>0 \Rightarrow d^2>4c^3$

(iii) Note that $c=-2, d=-2$ so

$$\begin{aligned} & & 0 &= x^3+6x-2 \\ \text{consider} & & 0 &= X^2-X-2 \\ & & &= (X+1)(X-2) \\ \Rightarrow & p=-1, q=2 & & \\ \Rightarrow & & 0 &= x^3-3\cdot 2\cdot(-1)x+2\cdot(-1)\cdot(-2+1) \\ \Rightarrow & & 0 &= -z^3+2 \\ \Rightarrow & & z &= \sqrt[3]{2} \\ \Rightarrow & & \frac{-z+2}{z+1} &= \sqrt[3]{2} \\ \Rightarrow & & -z+2 &= \sqrt[3]{2}z+\sqrt[3]{2} \\ \Rightarrow & & z &= \frac{2-\sqrt[3]{2}}{\sqrt[3]{2}+1} \end{aligned}$$

(iv)

$$\begin{aligned} & & 0 &= x^3-3p^2x+2p^3 \\ & & &= (x-p)(x^2+px-2p^2) \\ & & &= (x-p)^2(x+2p) \\ \Rightarrow & & x &= p, p, -2p \end{aligned}$$

Therefore if we have a repeated root to our associated quadratic we can find a cubic of the form $x^3-3p^2x+2p^3$, but we know this has roots we can find.

Question (2018 STEP III Q1) **(i)** The function f is given by

$$f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

Find the stationary point of the curve $y = f(\beta)$ and sketch the curve. Sketch also the curve $y = g(\beta)$, where

$$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

(ii) Let u and v be the roots of the equation

$$x^2 + \alpha x + \beta = 0\,,$$

where $\beta \neq 0$. Obtain expressions in terms of α and β for $u + v + \dfrac{1}{uv}$ and $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

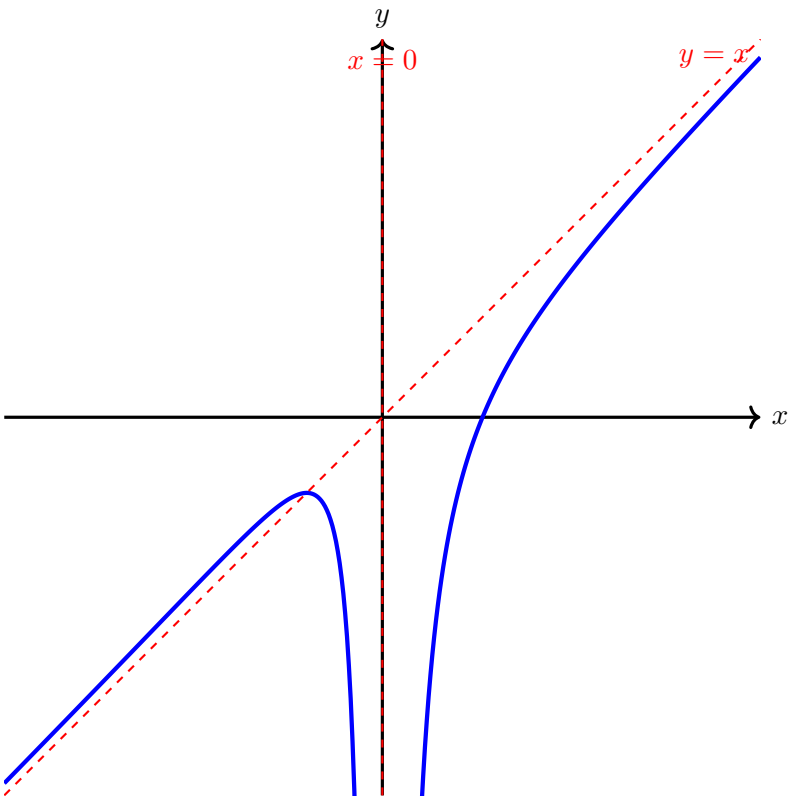
(iii) Given that $u+v+\dfrac{1}{uv} = -1$, and that u and v are real, show that $\dfrac{1}{u}+\dfrac{1}{v}+uv \leq -1$.

(iv) Given instead that $u+v+\dfrac{1}{uv} = 3$, and that u and v are real, find the greatest value of $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

(i)

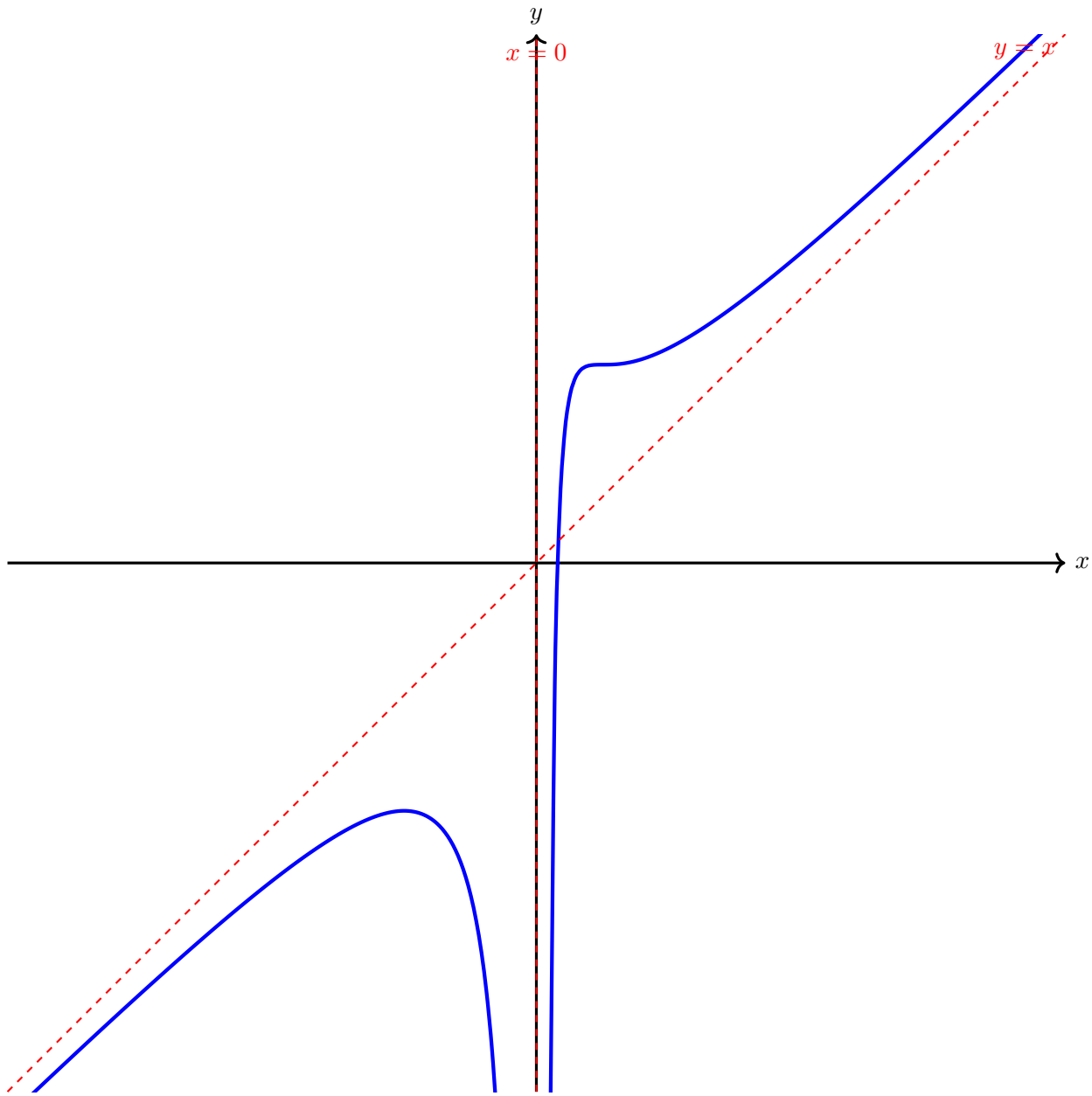
$$\begin{aligned}
& & f(\beta) &= \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \\
\Rightarrow & & f'(\beta) &= 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} \\
\Rightarrow & & 0 &= f'(\beta) \\
& & &= 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} \\
\Rightarrow & & 0 &= \beta^3 + \beta + 2 \\
& & &= (\beta + 1)(\beta^2 - \beta + 2)
\end{aligned}$$

Therefore the only stationary point is at $\beta = -1, f(-1) = -1$



$$
\begin{aligned}
& & g(\beta) &= \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \\
\Rightarrow & & g'(\beta) &= 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3} \\
\Rightarrow & & 0 &= f'(\beta) \\
& & &= 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3} \\
\Rightarrow & & 0 &= \beta^3 - 3\beta + 2 \\
& & &= (\beta - 1)^2(\beta + 2)
\end{aligned}
$$

Therefore there are stationary points at $\beta = 1, f(1) = 3, \beta = -2, f(-2) = \frac{1}{4}$



(ii) Let u, v be the roots of $x^2 + \alpha x + \beta = 0$, then since $(x-u)(x-v) = 0$ we must have $\alpha = -(u+v), \beta = uv$.

Therefore:

$$\begin{aligned} u + v + \frac{1}{uv} &= -\alpha + \frac{1}{\beta} \\ \frac{1}{u} + \frac{1}{v} + uv &= \frac{u+v}{uv} + uv \\ &= -\frac{\alpha}{\beta} + \beta \end{aligned}$$

Given $u + v + \frac{1}{uv} = -1$, ie $-\alpha + \frac{1}{\beta} = -1$. Since the roots are real, we must also have that $\alpha^2 - 4\beta \geq 0$, so

$$\begin{aligned} & -\alpha + \frac{1}{\beta} = -1 \\ \Rightarrow \quad & \alpha = 1 + \frac{1}{\beta} \\ \Rightarrow \quad & -\frac{\alpha}{\beta} + \beta = -\frac{1}{\beta}\left(1 + \frac{1}{\beta}\right) + \beta \\ & = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \end{aligned}$$

So we want to maximise $f(\beta)$ subject to $\alpha^2 - 4\beta \geq 0$

$$\begin{aligned} & 0 \leq \alpha^2 - 4\beta \\ & = \left(1 + \frac{1}{\beta}\right)^2 - 4\beta \\ & = 1 + \frac{2}{\beta} + \frac{1}{\beta^2} - 4\beta \\ \Leftrightarrow \quad & 0 \leq -4\beta^3 + \beta^2 + 2\beta + 1 \\ & = -(\beta - 1)(4\beta^2 + 3\beta + 1) \\ \Leftrightarrow \quad & \beta \leq 1 \end{aligned}$$

But we know $f(\beta) \leq -1$ on $(-\infty, 1]$ so we're done.

(iii) Given that $-\alpha + \frac{1}{\beta} = 3$ we have

$$\begin{aligned} & -\alpha + \frac{1}{\beta} = 3 \\ \Rightarrow \quad & \alpha = -3 + \frac{1}{\beta} \\ \Rightarrow \quad & -\frac{\alpha}{\beta} + \beta = -\frac{1}{\beta}\left(-3 + \frac{1}{\beta}\right) + \beta \\ & = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \end{aligned}$$

which we want to maximise, subject to:

$$\begin{aligned} & 0 \leq \alpha^2 - 4\beta \\ & = \left(-3 + \frac{1}{\beta}\right)^2 - 4\beta \\ & = 9 - \frac{6}{\beta} + \frac{1}{\beta^2} - 4\beta \\ \Leftrightarrow \quad & 0 \leq -4\beta^3 + 9\beta^2 - 6\beta + 1 \\ & = -(\beta - 1)^2(4\beta - 1) \end{aligned}$$

$$\Leftrightarrow \qquad \beta \leq \frac{1}{4}$$

Therefore the maximum will either be $f(-2) = \frac{1}{4}$ or $f(\frac{1}{4}) = -\frac{15}{4}$. Therefore the maximum is $\frac{1}{4}$

Question (2019 STEP III Q4)

The nth degree polynomial $\mathrm{P}(x)$ is said to be *reflexive* if:

(a) $\mathrm{P}(x)$ is of the form $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$ where $n \geq 1$;

(b) $a_1, a_2, \ldots, a_n$ are real;

(c) the n (not necessarily distinct) roots of the equation $\mathrm{P}(x) = 0$ are $a_1, a_2, \ldots, a_n$.

(i) Find all reflexive polynomials of degree less than or equal to 3.

(ii) For a reflexive polynomial with $n > 3$, show that

$$2a_2 = -a_2^2 - a_3^2 - \cdots - a_n^2.$$

Deduce that, if all the coefficients of a reflexive polynomial of degree n are integers and $a_n \neq 0$, then $n \leq 3$.

(iii) Determine all reflexive polynomials with integer coefficients.

(i) Suppose $n = 1$, then all polynomials are reflexive (since $x - a_1$ has the root a_1.

Suppose $n = 2$, then we want

$$\begin{aligned} x^2 - a_1x + a_2 &= (x - a_1)(x - a_2) \\ &= x^2 - (a_1 + a_2)x + a_1a_2 \\ \Rightarrow \qquad a_2 &= 0 \end{aligned}$$

So all polynomials of the form $x^2 - a_1x$ work and no others.

Suppose $n = 3$ then we want

$$\begin{aligned} x^3 - a_1x^2 + a_2x - a_3 &= (x - a_1)(x - a_2)(x - a_3) \\ &= x^3 - (a_1 + a_2 + a_3)x + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 \\ \Rightarrow \qquad a_2 + a_3 &= 0 \\ a_2a_3 &= a_2 \\ \Rightarrow \qquad -a_2^2 &= a_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad a_2 &= 0, -1 \\ -a_1 a_2^2 &= -a_2 \\ \Rightarrow \qquad a_2 &= 0, a_2 = 1/a_1 \end{aligned}$$

So we need either $x^3 - a_1 x$ or $(x+1)^2(x-1) = x^3 + x^2 - x - 1$

(ii) Suppose $n > 3$ then

$$\begin{aligned} \sum a_i^2 &= \left(\sum a_i\right)^2 - 2\sum_{i<j} a_i a_j \\ &= a_1^2 - 2a_2 \\ \Rightarrow \qquad 2a_2 &= a_1^2 - \sum a_i^2 \\ &= -a_2^2 - a_3^2 - \cdots - a_n^2 \end{aligned}$$

So $(a_2+1)^2 = 1 - a_3^2 - \cdots - a_n^2$ so if $a_n > 0$ (or any other $a_i, i > 2$ for that matter) then we must have $a_n = \pm 1, a_3, \ldots a_{n-1} = 0$, but if $a_n = \pm 1$ $x = 0$ is not a root. Therefore we must have a_0 and $a_i = 0$ for all $i > 3$

(iii) The only reflexive polynomials therefore must be $x^n - kx^{n-1}$ and $x^{n+3} + x^{n+2} - x^{n+1} - x^n$

Question (2025 STEP III Q6) **(i)** Let a, b and c be three non-zero complex numbers with the properties $a+b+c=0$ and $a^2+b^2+c^2=0$. Show that a, b and c cannot all be real. Show further that a, b and c all have the same modulus.

(ii) Show that it is not possible to find three non-zero complex numbers a, b and c with the properties $a+b+c=0$ and $a^3+b^3+c^3=0$.

(iii) Show that if any four non-zero complex numbers a, b, c and d have the properties $a+b+c+d=0$ and $a^3+b^3+c^3+d^3=0$, then at least two of them must have the same modulus.

(iv) Show, by taking $c=1$, $d=-2$ and $e=3$ that it is possible to find five real numbers a, b, c, d and e with distinct magnitudes and with the properties $a+b+c+d+e=0$ and $a^3+b^3+c^3+d^3+e^3=0$.

(i) If a, b, c were all real then $a^2 + b^2 + c^2 = 0 \Rightarrow a, b, c = 0$ but they are non-zero. Therefore they cannot all be real.

Since $(a+b+c)^2 = 0$ we must have $ab + bc + ca = 0$. Therefore a, b, c must satisfy $x^3 - abc = 0 \Rightarrow$ they all have the same modulus, since they are all cube roots of the same number.

(ii) Notice that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \Rightarrow abc=0$ but therefore they cannot all be non-zero.

(iii) Suppose $a+b+c+d=0$ then note that $a^2+b^2+c^2+d^2=(a+b+c+d)^2-2\sum_{sym} ab$ and

$a^3+b^3+c^3+d^3=(a+b+c+d)^3-3(a+b+c+d)(ab+ac+ad+bc+bd+cd)+3(abc+abd+acd+bcd) \Rightarrow abc+abd+acd+bcd=0$. Therefore a,b,c,d are roots of a polynomial of the form $x^4-kx^2+l=0$, but this means they must come in pairs with the same modulus.

(iv) Suppose $c=1,d=-2,e=3$ so $c+d+e=2$ and $c^3+d^3+e^3=1-8+27=20$, so we need to find a,b satisfying $a+b=-2, a^2+b^2=-20$, ie $4=(a+b)^2=-20+2ab \Rightarrow ab=12$, so we need the roots of $x^2+2x+12=0$ which clearly have different modulus.