Question (1988 STEP III Q8)

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. For $i = 1, 2$, and 3, let P_i be the point $(at_i^2, 2at_i)$, where t_1, t_2 and t_3 are all distinct. Let A_1 be the area of the triangle formed by the tangents at P_1, P_2 and P_3, and let A_2 be the area of the triangle formed by the normals at P_1, P_2 and P_3. Using the fact that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of

$$\tfrac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix},$$

show that $A_3 = (t_1 + t_2 + t_3)^2 A_1$. Deduce a necessary and sufficient condition in terms of t_1, t_2 and t_3 for the normals at P_1, P_2 and P_3 to be concurrent.

$\frac{dy}{dt} = 2a, \frac{dx}{dt} = 2at \Rightarrow \frac{dy}{dx} = \frac{1}{t}$.

Therefore the equation of the tangent will be $\frac{y-2at}{x-at^2} = \frac{1}{t} \Rightarrow y = \frac{1}{t}x + at$ and normal will be $\frac{y-2at}{x-at^2} = -t \Rightarrow y = t(at^2 - x + 2a)$.

The tangents will meet when:

$$\begin{cases} t_i y - x &= at_i^2 \\ t_j y - x &= at_j^2 \end{cases}$$

$$\begin{aligned} \Rightarrow \quad (t_i - t_j)y &= a(t_i - t_j)(t_i + t_j) \\ \Rightarrow \quad y &= a(t_i + t_j) \\ x &= at_i t_j \end{aligned}$$

The normals will meet when:

$$\begin{cases} y + t_i x &= at_i^3 + 2at_i \\ y + t_j x &= at_j^3 + 2at_j \end{cases}$$

$$\begin{aligned} \Rightarrow \quad (t_i - t_j)x &= a(t_i - t_j)(t_i^2 + t_i t_j + t_j^2 + 2) \\ \Rightarrow \quad x &= a(t_i^2 + t_i t_j + t_j^2 + 2) \\ y &= -at_i t_j (t_i + t_j) \end{aligned}$$

Therefore the area of our triangles will be:

$$\begin{aligned} \tfrac{1}{2}\det\begin{pmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \end{pmatrix} &= \frac{a^2}{2}\det\begin{pmatrix} t_1t_2 & (t_1+t_2) & 1 \\ t_2t_3 & (t_2+t_3) & 1 \\ t_3t_1 & (t_3+t_1) & 1 \end{pmatrix} \\ &= \frac{a^2}{2}\det\begin{pmatrix} t_1t_2 & (t_1+t_2) & 1 \\ t_2(t_3-t_1) & (t_3-t_1) & 0 \\ t_1(t_3-t_2) & (t_3-t_2) & 0 \end{pmatrix} \\ &= \frac{a^2}{2}|(t_2-t_1)(t_3-t_2)(t_1-t_3)| \end{aligned}$$

and

$$\tfrac{1}{2}\det\begin{pmatrix} a(t_1^2+t_1t_2+t_2^2+2) & -at_1t_2(t_1+t_2) & 1 \\ a(t_2^2+t_2t_3+t_3^2+2) & -at_2t_3(t_2+t_3) & 1 \\ a(t_3^2+t_3t_1+t_1^2+2) & -at_3t_1(t_3+t_1) & 1 \end{pmatrix} = \frac{a^2}{2}\det\begin{pmatrix} (t_1^2+t_1t_2+t_2^2+2) & -t_1t_2(t_1+t_2) & 1 \\ (t_2^2+t_2t_3+t_3^2+2) & -t_2t_3(t_2+t_3) & 1 \\ (t_3^2+t_3t_1+t_1^2+2) & -t_3t_1(t_3+t_1) & 1 \end{pmatrix}$$

$$= \frac{a^2}{2} \det \begin{pmatrix} (t_1^2 + t_1t_2 + t_2^2 + 2) & -t_1t_2(t_1 + t_2) \\ t_3^2 - t_1^2 + t_2(t_3 - t_1) & t_2(t_1^2 + t_1t_2 - t_2t_3 - t_3^2) \\ t_3^2 - t_2^2 + t_1(t_3 - t_2) & t_1(t_2^2 + t_2t_1 - t_1t_3 - t_3^2) \end{pmatrix}$$

$$= \frac{a^2}{2} \det \begin{pmatrix} (t_1^2 + t_1t_2 + t_2^2 + 2) & -t_1t_2(t_1 + t_2) \\ (t_3 - t_1)(t_3 + t_2 + t_1) & t_2(t_1 - t_3)(t_1 + t_3 + t_2 \\ (t_3 - t_2)(t_3 + t_2 + t_1) & t_1(t_2 - t_3)(t_1 + t_2 + t_3 \end{pmatrix}$$

$$= \frac{a^2}{2}(t_1 + t_2 + t_3)^2 |(t_2 - t_1)(t_3 - t_2)(t_1 - t_3)|$$

as required.

The normals will be concurrent iff the area of their triangle is 0. This is certainly true if $t_1 + t_2 + t_3 = 0$. In fact the only if is also true, since no 3 tangents can be concurrent.

Question (1992 STEP III Q9)

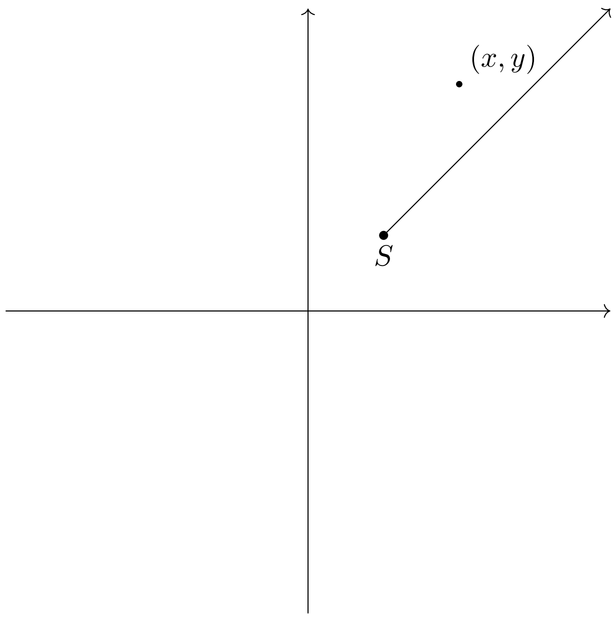
The straight line OSA, where O is the origin, bisects the angle between the positive x and y axes. The ellipse E has S as focus. In polar coordinates with S as pole and SA as the initial line, E has equation $\ell = r(1 + e\cos\theta)$. Show that, at the point on E given by $\theta = \alpha$, the gradient of the tangent to the ellipse is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\alpha - \cos\alpha - e}{\sin\alpha + \cos\alpha + e}.$$

The points on E given by $\theta = \alpha$ and $\theta = \beta$ are the ends of a diameter of E. Show that

$$\tan(\alpha/2)\tan(\beta/2) = -\frac{1+e}{1-e}.$$

[**Hint.** A diameter of an ellipse is a chord through its centre.]



$$\ell = r(1 + e\cos\theta)$$

$$\Rightarrow \quad 0 = \frac{\mathrm{d}r}{\mathrm{d}\theta}(1 + e\cos\theta) - re\sin\theta$$

$$\Rightarrow \qquad \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{re\sin\theta}{1+e\cos\theta}$$

Suppose we consider the (x', y') plane, which is essentially the $x - y$ plan rotated by $45°$, then we would have

$$\begin{aligned}
\frac{\mathrm{d}y'}{\mathrm{d}x'} &= \frac{\frac{\mathrm{d}y'}{\mathrm{d}\theta}}{\frac{\mathrm{d}x'}{\mathrm{d}\theta}} \\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \\
&= \frac{\frac{re\sin\theta}{1+e\cos\theta}\sin\theta + r\cos\theta}{\frac{re\sin\theta}{1+e\cos\theta}\cos\theta - r\sin\theta} \\
&= \frac{re\sin^2\theta + r\cos\theta(1+e\cos\theta)}{re\sin\theta\cos\theta - r\sin\theta(1+e\cos\theta)} \\
&= \frac{\cos\theta + e\cos^2\theta + e\sin^2\theta}{-\sin\theta} \\
&= \frac{\cos\theta + e}{-\sin\theta}
\end{aligned}$$

Since our frame is rotated by $45°$ we need to consider the appropriate gradient for this. We know that $m = \tan\theta$ so $m' = \tan(\theta + 45°) = \frac{1+m}{1-m}$
therefore we should have

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 + \frac{\cos\theta+e}{-\sin\theta}}{1 - \frac{\cos\theta+e}{-\sin\theta}} \\
&= \frac{\cos\theta - \sin\theta + e}{-\sin\theta - \cos\theta - e} \\
&= \frac{\sin\theta - \cos\theta - e}{\sin\theta + \cos\theta + e}
\end{aligned}$$

As required.
The tangents at those points are parallel, therefore

$$\begin{aligned}
\frac{\cos\alpha + e}{\sin\alpha} &= \frac{\cos\beta + e}{\sin\beta} \\
\Rightarrow \qquad \frac{\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + e}{\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}} &= \frac{\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} + e}{\frac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}} \\
\frac{1 - \tan^2\frac{\alpha}{2} + e(1+\tan^2\frac{\alpha}{2})}{2\tan\frac{\alpha}{2}} &= \frac{1 - \tan^2\frac{\beta}{2} + e(1+\tan^2\frac{\beta}{2})}{2\tan\frac{\beta}{2}} \\
\frac{(1+e) + (e-1)\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}} &= \frac{(1+e) + (e-1)\tan^2\frac{\beta}{2}}{2\tan\frac{\beta}{2}} \\
\frac{(1+e)}{\tan\frac{\alpha}{2}} - (1-e)\tan\frac{\alpha}{2} &= \frac{(1+e)}{\tan\frac{\beta}{2}} - (1-e)\tan\frac{\beta}{2}
\end{aligned}$$

ie both $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of a quadratic of the form $(1-e)x^2 - cx - (1+e)$ but this means the product of the roots is $-\frac{1+e}{1-e}$

Question (1994 STEP I Q5)

A parabola has the equation $y = x^2$. The points P and Q with coordinates (p, p^2) and (q, q^2) respectively move on the parabola in such a way that $\angle POQ$ is always a right angle.

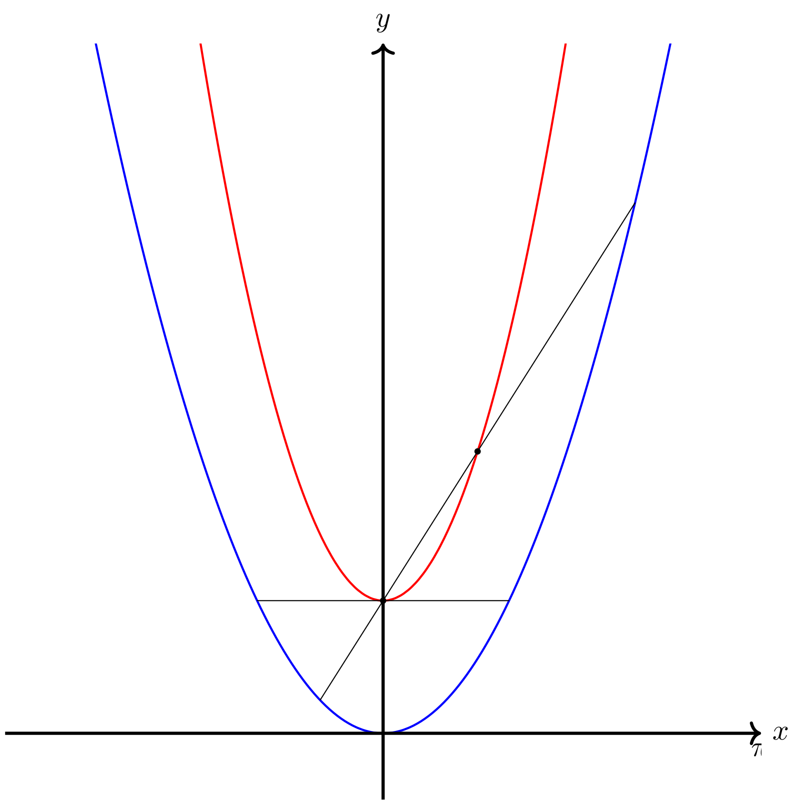
(i) Find and sketch the locus of the midpoint R of the chord PQ.

(ii) Find and sketch the locus of the point T where the tangents to the parabola at P and Q intersect.

(i) The line PO has gradient $\frac{p^2}{p} = p$ and teh line QO has gradient q, therefore we must have that $pq = -1$. Therefore, R is the point

$$\begin{aligned} R &= \left(\frac{p - \frac{1}{p}}{2}, \frac{p^2 + \frac{1}{p^2}}{2} \right) \\ &= \left(\frac{1}{2}\left(p - \frac{1}{p}\right), 2\left(\frac{1}{2}\left(p - \frac{1}{p}\right)\right)^2 + 1 \right) \\ &= \left(t, 2t^2 + 1\right) \end{aligned}$$

So we are looking at another parabola.

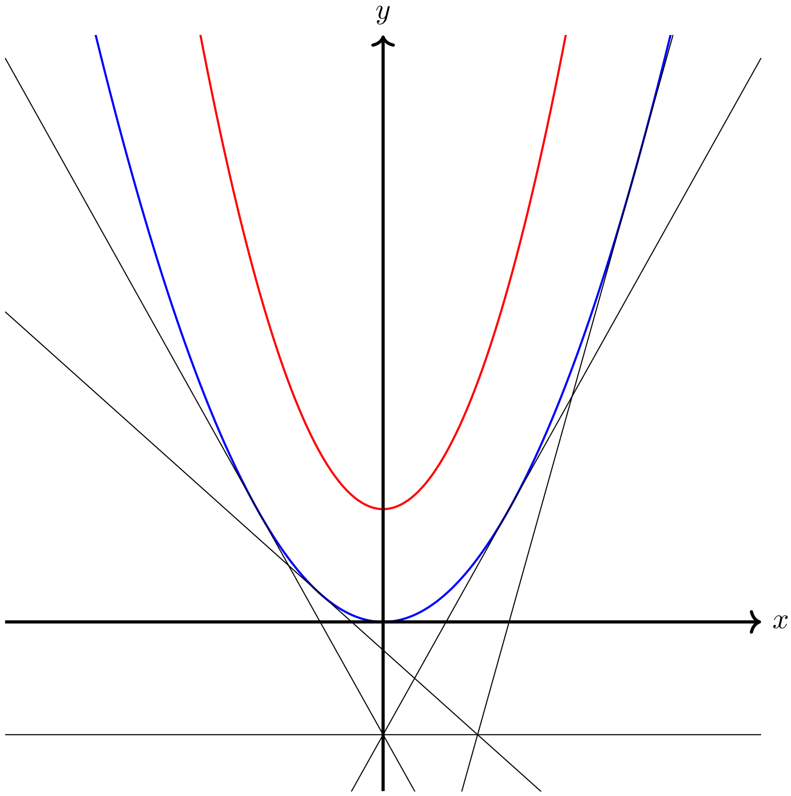


(ii) The tangents are $y = 2px + c$, ie $p^2 = 2p^2 + c$, ie $y = 2px - p^2$ so we have

$$y - 2px = -p^2$$

$$
\begin{aligned}
& & y - 2qx &= -q^2 \\
&\Rightarrow & (2p-2q)x &= p^2 - q^2 \\
&\Rightarrow & x &= \frac{1}{2}(p+q) \\
& & y &= p(p+q) - p^2 \\
& & y &= pq = -1
\end{aligned}
$$

Therefore $x = \frac{1}{2}(p - \frac{1}{p}), y = -1$, so we have the line $y = -1$ (the directrix)



Question (2003 STEP III Q7)

In the x–y plane, the point A has coordinates $(a\,,0)$ and the point B has coordinates $(0\,,b)$, where a and b are positive. The point P, which is distinct from A and B, has coordinates (s,t). X and Y are the feet of the perpendiculars from P to the x–axis and y–axis respectively, and N is the foot of the perpendicular from P to the line AB. Show that the coordinates $(x\,,y)$ of N are given by

$$x = \frac{ab^2 - a(bt - as)}{a^2 + b^2}\,, \quad y = \frac{a^2b + b(bt - as)}{a^2 + b^2}\,.$$

Show that, if $\left(\dfrac{t-b}{s}\right)\left(\dfrac{t}{s-a}\right) = -1$, then N lies on the line XY.

Give a geometrical interpretation of this result.

Question (2005 STEP III Q5)

Let P be the point on the curve $y = ax^2 + bx + c$ (where a is non-zero) at which the gradient is m. Show that the equation of the tangent at P is

$$y - mx = c - \frac{(m-b)^2}{4a}\,.$$

Show that the curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ (where a_1 and a_2 are non-zero) have a common tangent with gradient m if and only if

$$(a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + 4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2 = 0\,.$$

Show that, in the case $a_1 \neq a_2$, the two curves have exactly one common tangent if and only if they touch each other. In the case $a_1 = a_2$, find a necessary and sufficient condition for the two curves to have exactly one common tangent.

$$\begin{aligned} && y' &= 2ax + b \\ \Rightarrow && m &= 2ax_t + b \\ \Rightarrow && x_t &= \frac{m-b}{2a} \end{aligned}$$

Therefore we must have

$$\begin{aligned} mx_t &= 2ax_t^2 + bx_t \\ y - mx &= ax_t^2 + bx_t + c - mx_t \\ &= ax_t^2 + bx_t + c - (2ax_t^2 + bx_t) \\ &= c - ax_t^2 \\ &= c - a\left(\frac{m-b}{2a}\right)^2 \\ &= c - \frac{(m-b)^2}{4a} \end{aligned}$$

They will have a common tangent if and only if the constant terms are equal, ie

$$\begin{aligned} && c_1 - \frac{(m-b_1)^2}{4a_1} &= c_2 - \frac{(m-b_2)^2}{4a_2} \\ \Leftrightarrow && (c_1 - c_2) &= \frac{(m-b_1)^2}{4a_1} - \frac{(m-b_2)^2}{4a_2} \\ \Leftrightarrow && 4a_1a_2(c_1 - c_2) &= a_2(m-b_1)^2 - a_1(m-b_2)^2 \\ && &= (a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + a_2b_1^2 - a_1b_2^2 \end{aligned}$$

as required.

Treating this as a polynomial in m, we can see that the two curves will have exactly one common tangent iff $\Delta = 0$, ie:

$$\begin{aligned} 0 &= \Delta \\ &= (2(a_1b_2 - a_2b_1))^2 - 4(a_2 - a_1)(4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2) \end{aligned}$$

$$= 4a_1^2b_2^2 - 8a_1a_2b_1b_2 + 4a_2b_1^2 - 4a_2^2b_1^2 - 4a_1^2b_2^2 + 4a_1a_2(b_1^2 + b_2^2) - 16(a_2 - a_1)a_1a_2(c_2 - c_1)$$
$$= -8a_1a_2b_1b_2 + 4a_1a_2(b_1^2 + b_2^2) - 16(a_2 - a_1)a_1a_2(c_2 - c_1)$$
$$= a_1a_2(4(b_1 - b_2)^2 - 16(a_2 - a_1)(c_2 - c_1))$$
$$= 4a_1a_2((b_2 - b_1)^2 - 4(a_2 - a_1)(c_2 - c_1)$$

But this is just the discriminant of the difference, ie equivalent to the two parabolas just touching. (Assuming $a_1 - a_2 \neq 0$ and we do end up with a quadratic).

If $a_1 = a_2 = a$ then we need exactly one solution to $2a(b_1 - b_2)m + 4a^2(c_2 - c_1) + a(b_1^2 - b_2^2) = 0$, ie $b_1 \neq b_2$.

Question (2008 STEP III Q3)

The point $P(a\cos\theta\,,\, b\sin\theta)$, where $a > b > 0$, lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

The point $S(-ea\,,\,0)$, where $b^2 = a^2(1-e^2)$, is a focus of the ellipse. The point N is the foot of the perpendicular from the origin, O, to the tangent to the ellipse at P. The lines SP and ON intersect at T. Show that the y-coordinate of T is

$$\frac{b\sin\theta}{1+e\cos\theta}\,.$$

Show that T lies on the circle with centre S and radius a.

Find the gradient of the tangent of the ellipse at P:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xb^2}{2ya^2}$$
$$= -\frac{a\cos\theta b^2}{b\sin\theta a^2}$$
$$= -\frac{b}{a}\cot\theta$$

Therefore the gradient of ON is $\frac{a}{b}\tan\theta$.

$$y = \frac{a}{b}\tan\theta x$$
$$\frac{y-0}{x-(-ea)} = \frac{b\sin\theta - 0}{a\cos\theta - (-ea)}$$
$$y = \frac{b\sin\theta}{a(e+\cos\theta)}(x+ea)$$
$$\Rightarrow \quad y = \frac{b\sin\theta}{a(\cos\theta+e)}\frac{b}{a}\cot\theta y + \frac{eb\sin\theta}{\cos\theta+e}$$
$$= \frac{b^2\cos\theta}{a^2(\cos\theta+e)}y + \frac{eb\sin\theta}{\cos\theta+e}$$
$$\Rightarrow \quad (\cos\theta+e)y = (1-e^2)\cos\theta y + eb\sin\theta$$
$$e(1+e\cos\theta)y = eb\sin\theta$$

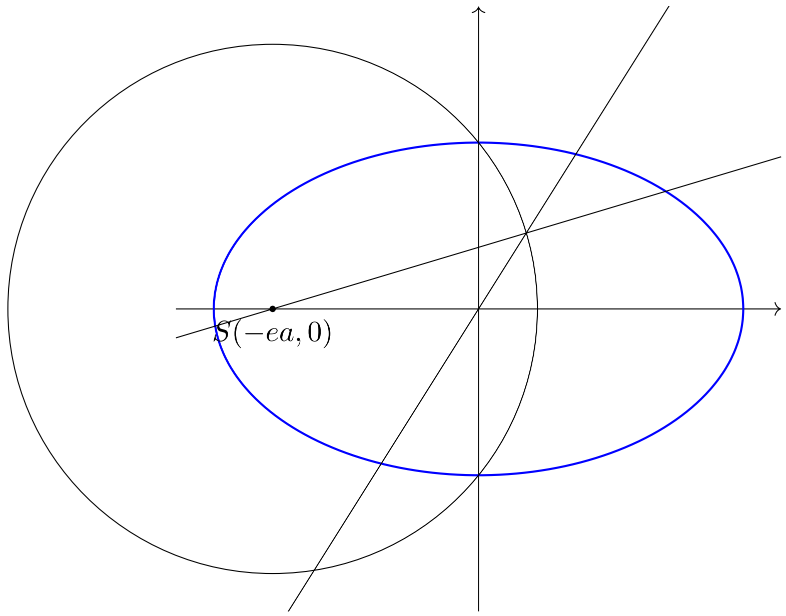
$$\Rightarrow \qquad y = \frac{b\sin\theta}{1+e\cos\theta}$$

$$x = \frac{b\sin\theta}{1+e\cos\theta}\frac{b}{a}\cot\theta = \frac{b^2\cos\theta}{a(1+e\cos\theta)}$$

Therefore $T\left(\dfrac{b^2\cos\theta}{a(1+e\cos\theta)}, \dfrac{b\sin\theta}{1+e\cos\theta}\right)$.
Finally, we can look at the distance of T from S

$$\begin{aligned}
d^2 &= \left(\frac{b^2\cos\theta}{a(1+e\cos\theta)} - (-ea)\right)^2 + \left(\frac{b\sin\theta}{1+e\cos\theta} - 0\right)^2 \\
&= \frac{\left(b^2\cos\theta + ea^2(1+e\cos\theta)\right)^2 + (ab\sin\theta)^2}{a^2(1+e\cos\theta)^2} \\
&= \frac{b^4\cos^2\theta + e^2a^4(1+e\cos\theta)^2 + 2ea^2b^2(1+e\cos\theta) + a^2b^2\sin^2\theta}{a^2(1+e\cos\theta)^2} \\
&= \frac{a^4(1-e^2)^2\cos^2\theta + e^2a^4(1+e\cos\theta)^2 + 2ea^2a^2(1-e^2)(1+e\cos\theta) + a^4(1-e^2)\sin^2\theta}{a^2(1+e\cos\theta)^2} \\
&= a^2\left(\frac{(1-e^2)^2\cos^2\theta + e^2(1+e\cos\theta)^2 + 2e(1-e^2)(1+e\cos\theta) + (1-e^2)(1-\cos^2\theta)}{(1+e\cos\theta)^2}\right) \\
&= a^2\left(\frac{e^2(1+e\cos\theta)^2 + (1-e^2)((1-e^2)\cos^2\theta + 2e(1+e\cos\theta) + (1-\cos^2\theta))}{(1+e\cos\theta)^2}\right) \\
&= a^2\left(\frac{e^2(1+e\cos\theta)^2 + (1-e^2)(1+e\cos\theta)^2}{(1+e\cos\theta)^2}\right) \\
&= a^2
\end{aligned}$$

Therefore a circle radius a centre S.



Question (2014 STEP III Q3) **(i)** The line L has equation $y = mx + c$, where $m > 0$ and $c > 0$. Show that, in the case $mc > a > 0$, the shortest distance between L and the parabola $y^2 = 4ax$ is

$$\frac{mc - a}{m\sqrt{m^2+1}}.$$

What is the shortest distance in the case that $mc \leq a$?

(ii) Find the shortest distance between the point $(p, 0)$, where $p > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the value of p/a. [*You may wish to use the parametric coordinates* $(at^2, 2at)$ *of points on the parabola.*] Hence find the shortest distance between the circle $(x-p)^2 + y^2 = b^2$, where $p > 0$ and $b > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the values of p, a and b.

Question (2016 STEP III Q2)

The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P.

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point that is independent of the choice of P.

(iii) Let T be the point of intersection of OP and QR, where O is the coordinate origin. Show that T lies on a line that is independent of the choice of P. Show further that the distance from the x-axis to T is less than $\dfrac{a}{\sqrt{2}}$.

(i)

$$\begin{aligned} 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 4a \\ \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2a}{y} \end{aligned}$$

Therefore we must have

$$\begin{aligned} \underbrace{-\frac{2aq}{2a}}_{\text{gradient of normal}} &= \underbrace{\frac{2ap-2aq}{ap^2-aq^2}}_{\Delta y/\Delta x} \\ \Rightarrow \quad -q &= \frac{2}{p+q} \\ 0 &= 2 + pq + q^2 \end{aligned}$$

(ii) We must have that q, r are the two roots of $x^2 + px + 2 = 0$

QR has the equation:

$$\begin{aligned}
\frac{y - 2aq}{x - aq^2} &= \frac{2ar - 2aq}{ar^2 - aq^2} \\
\Rightarrow \qquad \frac{y - 2aq}{x - aq^2} &= \frac{2}{r + q} \\
\Rightarrow \qquad y &= \frac{2}{q + r}(x - aq^2) + 2aq \\
y &= -\frac{2}{p}x + 2a\left(q - \frac{q^2}{q + r}\right) \\
y &= -\frac{2}{p}x + 2a\frac{qr}{q + r} \\
y &= -\frac{2}{p}x - 2a\frac{2}{p} \\
y &= -\frac{2}{p}(x + 2a)
\end{aligned}$$

Therefore the point $(-2a, 0)$ lies on all such lines.

(iii) OP has equation $y = \frac{2}{p}x$

$$\begin{aligned}
y &= \frac{2}{p}x \\
y &= -\frac{2}{p}(x + 2a) \\
2y &= -\frac{4a}{p} \\
\Rightarrow \qquad y &= -\frac{2a}{p} \\
x &= -a
\end{aligned}$$

Therefore $T\left(-a, -\frac{2a}{p}\right)$ always lies on the line $x = -a$

The distance to the x-axis from T is $\frac{2a}{|p|}$. We need to show that p can't be too small. Specifically $x^2 + px + 2 = 0$ must have 2 real roots, ie $\Delta = p^2 - 8 \geq 0 \Rightarrow |p| \geq 2\sqrt{2}$, ie $\frac{2a}{|p|} \leq \frac{2a}{2\sqrt{2}} = \frac{a}{\sqrt{2}}$ as required.