(i) Notice that $f(x) = x - \tanh x$ has $f'(x) = 1 - \text{sech}^2 x = \tanh^2 x > 0$ so $f(x)$ is strictly increasing on $(0, \infty)$ and $f(0) = 0$ therefore $f(x)$ is positive for all x positive

(ii) Let $f(x) = x\sinh x - 2\cosh x + 2$ then $f'(x) = \sinh x + x\cosh x - 2\sinh x = x\cosh x - \sinh x = \cosh x(x - \tanh x) > 0$ by the first part. $f(0) = 0$ so $f(x)$ is positive for all x positive.

(iii) Let $f(x) = 2x\cosh 2x - 3\sinh 2x + 4x$ then

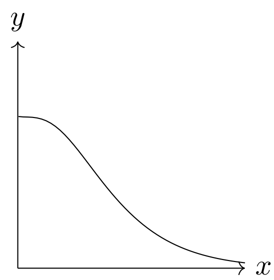
$$\begin{aligned} f'(x) &= 2\cosh 2x + 4x\sinh 2x - 6\cosh 2x + 4 \\ &= 4(x\sinh 2x - \cosh 2x + 1) \\ &= 4(x 2\cosh x \sinh x - 2\cosh^2 x) \\ &= 8\cosh^2 x(x - \tanh x) \end{aligned}$$

Which is always positive when x ¿ 0, $f(0) = 0$ so $f(x) > 0$ for all positive x.

Let $f(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}$ then

$$\begin{aligned} f'(x) &= \frac{(\cosh x)^{\frac{1}{3}}\sinh x + \frac{1}{3}x\cosh^{-\frac{2}{3}} x\sinh^2 x - x(\cosh x)^{\frac{1}{3}}\cosh x}{\sinh^2 x} \\ &= \frac{\cosh x\sinh x + \frac{1}{3}x\sinh^2 x - x\cosh^2 x}{\cosh x^{\frac{2}{3}} x\sinh^2 x} \\ &= \frac{3\cosh x\sinh x + x(\sinh^2 x - 3\cosh^2 x)}{3\cosh x^{\frac{2}{3}} x\sinh^2 x} \\ &= \frac{\frac{3}{2}\sinh 2x + x(-2\cosh 2x - 2)}{3\cosh x^{\frac{2}{3}} x\sinh^2 x} \\ &= \frac{3\sinh 2x - 4x\cosh 2x - 4x}{6\cosh x^{\frac{2}{3}} x\sinh^2 x} \end{aligned}$$

which from the earlier part is always negative.



Question (1989 STEP II Q3)

The real numbers x and y are related to the real numbers u and v by

$$2(u + \mathrm{i}v) = \mathrm{e}^{x+\mathrm{i}y} - \mathrm{e}^{-x-\mathrm{i}y}.$$

Show that the line in the x-y plane given by $x = a$, where a is a positive constant, corresponds to the ellipse

$$\left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2 = 1$$

in the u-v plane. Show also that the line given by $y = b$, where b is a constant and $0 < \sin b < 1$, corresponds to one branch of a hyperbola in the u-v plane. Write down the u and v coordinates of one point of intersection of the ellipse and hyperbola branch, and show that the curves intersect at right-angles at this point. Make a sketch of the u-v plane showing the ellipse, the hyperbola branch and the line segments corresponding to:

(i) $x = 0$;

(ii) $y = \frac{1}{2}\pi, \quad 0 \leqslant x \leqslant a$.

$$\begin{aligned}
2(u+iv) &= e^{a+iy} - e^{-a-iy} \\
&= (e^a \cos y - e^{-a}\cos y) + (e^a \sin y + e^{-a}\sin y)i \\
&= 2\sinh a \cos y + 2\cosh a \sin y i \\
\Rightarrow \qquad \frac{u}{\sinh a} &= \cos y \\
\frac{v}{\cosh a} &= \sin y \\
\Rightarrow \qquad 1 &= \left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2
\end{aligned}$$

$$\begin{aligned}
2(u+iv) &= e^{x+ib} - e^{-x-ib} \\
&= 2\sinh x \cos b + 2\cosh x \sin b i \\
\Rightarrow \qquad \frac{u}{\cos b} &= \sinh x \\
\frac{v}{\sin b} &= \cosh x \\
\Rightarrow \qquad 1 &= \left(\frac{v}{\sin b}\right)^2 - \left(\frac{u}{\cos b}\right)^2
\end{aligned}$$

Therefore all the points lie of a hyperbola, and since $\frac{v}{\sin b} > 0 \Rightarrow v > 0$ it's one branch of the hyperbola. (And all points on it are reachable as x varies from $-\infty < x < \infty$.

$$\begin{aligned}
2(u+iv) &= e^{a+ib} - e^{-a-ib} \\
&= 2\sinh a \cos b + 2\cosh a \sin b i
\end{aligned}$$

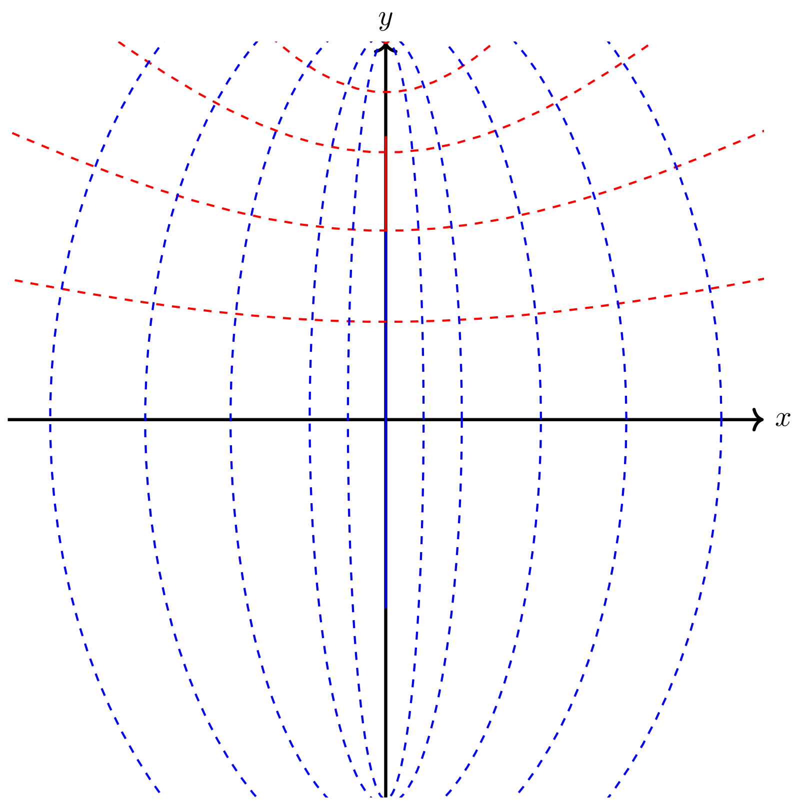
so we can take $u = \sinh a \cos b, v = \cosh a \sin b$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} \quad & 0 = \frac{2u}{\sinh^2 a} + \frac{2v}{\cosh^2 a}\frac{\mathrm{d}v}{\mathrm{d}u} \\
\Rightarrow \quad & \frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{u}{v}\coth^2 a
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}u}|_{(u,v)} &= -\frac{\sinh a \cos b}{\cosh a \sin b}\coth^2 a \\
&= -\cot b \coth a
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} \quad & 0 = \frac{2v}{\sin^2 b}\frac{\mathrm{d}v}{\mathrm{d}u} - \frac{2u}{\cos^2 b} \\
\Rightarrow \quad & \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u}{v}\tan^2 b \\
& \frac{\mathrm{d}v}{\mathrm{d}u}|_{(u,v)} = \frac{\sinh a \cos b}{\cosh a \sin b}\tan^2 b \\
& = \tanh a \tan b
\end{aligned}$$

Therefore they are negative reciprocals and hence perpendicular.



Question (1989 STEP III Q5)

Given that $y = \cosh(n\cosh^{-1} x)$, for $x \geqslant 1$, prove that

$$y = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

Explain why, when $n = 2k+1$ and $k \in \mathbb{Z}^+$, y can also be expressed as the polynomial

$$a_0 x + a_1 x^3 + a_2 x^5 + \cdots + a_k x^{2k+1}.$$

Find a_0, and show that

(i) $a_1 = (-1)^{k-1} 2k(k+1)(2k+1)/3$;

(ii) $a_2 = (-1)^k 2(k-1)k(k+2)(2k+1)/15$.

Find also the value of $\displaystyle\sum_{r=0}^{k} a_r$.

Recall, $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

$$\begin{aligned}
\cosh(n\cosh^{-1} x) &= \frac{1}{2}\left(\exp(n\cosh^{-1} x) + \exp(-n\cosh^{-1} x)\right) \\
&= \frac{1}{2}\left((x + \sqrt{x^2-1})^n + (x + \sqrt{x^2-1})^{-n}\right) \\
&= \frac{1}{2}\left((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right)
\end{aligned}$$

When $n = 2k + 1$

$$\begin{aligned}
\cosh(n\cosh^{-1} x) &= \frac{1}{2}\left((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right) \\
&= \frac{1}{2}\left(\sum_{i=0}^{2k+1} \binom{2k+1}{i} x^{2k+1-i} \left((\sqrt{x^2-1}^{\,i} + (-\sqrt{x^2-1})^i\right)\right) \\
&= \sum_{i=0}^{k} \binom{2k+1}{2i} x^{2k+1-2i} (x^2-1)^i \\
&= \sum_{i=0}^{k} \binom{2k+1}{2i} x^{2(k-i)+1} (x^2-1)^i
\end{aligned}$$

Which is clearly a polynomial with only odd degree terms.

$$\begin{aligned}
a_0 &= \frac{\mathrm{d}y}{\mathrm{d}x}|_{x=0} \\
&= \sum_{i=0}^{k} \binom{2k+1}{2i} \left((2(k-i)+1)x^{2(k-i)}(x^2-1)^i + 2i \cdot x^{2(k-i)+2}(x^2-1)\right)
\end{aligned}$$

$$= \binom{2k+1}{2k}(-1)^k$$
$$= (-1)^k(2k+1)$$

(i)

$$\begin{aligned}
a_1 &= \binom{2k+1}{2k}\binom{k}{1}(-1)^{k-1} + \binom{2k+1}{2(k-1)}(-1)^{k-1} \\
&= (-1)^{k-1} \cdot ((2k+1)k + \frac{(2k+1)\cdot 2k \cdot (2k-1)}{3!}) \\
&= (-1)^{k-1}(2k+1)k\frac{3+2k-1}{3} \\
&= (-1)^{k-1}2(2k+1)k(k+1)
\end{aligned}$$

(ii)

$$\begin{aligned}
a_2 &= \binom{2k+1}{2k}\binom{k}{2}(-1)^{k-2} + \binom{2k+1}{2(k-1)}\binom{k-1}{1}(-1)^{k-2} + \binom{2k+1}{2(k-2)}(-1)^{k-2} \\
&= \binom{2k+1}{1}\binom{k}{2}(-1)^{k-2} + \binom{2k+1}{3}\binom{k-1}{1}(-1)^{k-2} + \binom{2k+1}{5}(-1)^{k-2} \\
&= (-1)^k\left(\binom{2k+1}{1}\frac{k(k-1)}{2} + \binom{2k+1}{3}(k-1) + \binom{2k+1}{5}\right) \\
&= (-1)^k\left(\frac{(2k+1)k(k-1)}{2} + \frac{(2k+1)k(2k-1)}{3} + \frac{(2k+1)k(2k-1)(k-1)(2k-3)}{5\cdot 2\cdot 3}\right) \\
&= (-1)^k(2k+1)k\frac{1}{30}\left(15(k-1) + 10(2k-1) + (2k-1)(k-1)(2k-3)\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{r=0}^{k} a_k &= \frac{1}{2}\left((1+\sqrt{1^2-1})^n + (1-\sqrt{1^2-1})^n\right) \\
&= 1
\end{aligned}$$

Question (1990 STEP III Q9)

The real variables θ and u are related by the equation $\tan\theta = \sinh u$ and $0 \leqslant \theta < \frac{1}{2}\pi$. Let $v = \text{sech} u$. Prove that

(i) $v = \cos\theta$;

(ii) $\dfrac{\mathrm{d}\theta}{\mathrm{d}u} = v$;

(iii) $\sin 2\theta = -2\dfrac{\mathrm{d}v}{\mathrm{d}u}$ and $\cos 2\theta = -\cosh u\dfrac{\mathrm{d}^2 v}{\mathrm{d}u^2}$;

(iv) $\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\dfrac{\mathrm{d}^2 v}{\mathrm{d}\theta^2} + \dfrac{\mathrm{d}v}{\mathrm{d}\theta}\dfrac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \left(\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = 0.$

(i)

$$
\begin{aligned}
v &= \text{sech} u \\
&= \frac{1}{\text{cosh} u} \\
&= \frac{1}{\sqrt{1+\text{sinh}^2 u}} && (u > 0) \\
&= \frac{1}{\sqrt{1+\tan^2\theta}} \\
&= \frac{1}{\sqrt{\text{sec}^2\theta}} \\
&= \cos\theta && (0 < \theta < \tfrac{\pi}{2})
\end{aligned}
$$

(ii)

$$
\begin{aligned}
& & \tan\theta &= \text{sinh} u \\
&\underbrace{\Rightarrow}_{\frac{\mathrm{d}}{\mathrm{d}u}} & \sec^2\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} &= \cosh u \\
&\Rightarrow & \frac{\mathrm{d}\theta}{\mathrm{d}u} &= \cosh u\cdot\cos^2\theta \\
& & &= \frac{1}{v}\cdot v^2 \\
& & &= v
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= 2\sin\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} \\
&= -2\frac{\mathrm{d}v}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} && (\cos\theta = v) \\
&= -2\frac{\mathrm{d}v}{\mathrm{d}u}
\end{aligned}
$$

$$
\begin{aligned}
& & \sin 2\theta &= -2\frac{\mathrm{d}v}{\mathrm{d}u} \\
&\underbrace{\Rightarrow}_{\frac{\mathrm{d}}{\mathrm{d}u}} & 2\cos 2\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} &= -2\frac{\mathrm{d}^2v}{\mathrm{d}u^2} \\
&\Rightarrow & \cos 2\theta &= -\frac{\mathrm{d}^2v}{\mathrm{d}u^2}\frac{1}{v} \\
& & &= -\frac{\mathrm{d}^2v}{\mathrm{d}u^2}\cosh u
\end{aligned}
$$

(iv)

$$
\frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{1}{v}
$$

$$\Rightarrow \quad \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}\theta}$$
$$= \frac{1}{v^2}\sin\theta$$
$$\frac{\mathrm{d}v}{\mathrm{d}\theta} = -\sin\theta$$
$$\Rightarrow \quad \frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} = -\cos\theta$$
$$= -v$$

Therefore

$$\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} + \frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \frac{1}{v}\cdot(-v) + (-\sin\theta)\cdot\left(\frac{1}{v^2}\sin\theta\right) + \frac{1}{v^2}$$
$$= -1 + \frac{1-\sin^2\theta}{v^2}$$
$$= -1 + \frac{\cos^2\theta}{v^2}$$
$$= -1 + 1$$
$$= 0$$

Question (1991 STEP II Q8)

Solve the quadratic equation $u^2 + 2u\sinh x - 1 = 0$, giving u in terms of x. Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

which satisfies $y = 0$ and $y' > 0$ at $x = 0$. Find the solution of the differential equation

$$\sinh x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x = 0$$

which satisfies $y = 0$ at $x = 0$.

$$0 = u^2 + 2u\sinh x - 1$$
$$= u^2 + u(e^x - e^{-x}) - e^x e^{-x}$$
$$= (u - e^{-x})(u + e^x)$$
$$\Rightarrow \quad u = e^{-x}, -e^x$$

$$0 = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1$$
$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}, -e^x$$
$$\Rightarrow \quad y = -e^{-x} + C, -e^x + C$$

$$\begin{aligned} y(0) = 0: & \qquad C = 1 \text{ both cases} \\ y'(0) > 0: & \qquad y = 1 - e^{-x} \end{aligned}$$

$$\begin{aligned} 0 &= \sinh x u^2 + 2u - \sinh x \\ \Rightarrow \qquad u &= \frac{-2 \pm \sqrt{4 + 4\sinh^2 x}}{2\sinh x} \\ &= \frac{-1 \pm \cosh x}{\sinh x} = -\text{cosech } x \pm \coth x \end{aligned}$$

$$\begin{aligned} 0 &= \sinh x \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x \\ \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} &= -\text{cosech } x \pm \coth x \\ \Rightarrow \qquad y &= -\ln\left(\tanh\frac{x}{2}\right) \pm \ln\sinh x + C \end{aligned}$$

For $x \to 0$ to be defined, we need $+$, so

$$\begin{aligned} y &= \ln\left(\frac{\sinh x}{\tanh\frac{x}{2}}\right) + C \\ y &= \ln\left(\frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\tanh\frac{x}{2}}\right) + C \\ &= \ln\left(2\cosh^2 x\right) + C \\ y(0) = 0: \qquad 0 &= \ln 2 + C \\ \Rightarrow \qquad y &= \ln(2\cosh^2 x) - \ln 2 \\ y &= 2\ln(\cosh x) \end{aligned}$$

Question (1991 STEP III Q6)

The transformation T from $\binom{x}{y}$ to $\binom{x'}{y'}$ in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where u is a positive real constant. Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves. A line, not necessary through the origin, which has gradient $\tanh v$ transforms under T into a line with gradient $\tanh v'$. Show that $v' = v + u$. The lines ℓ_1 and ℓ_2 with gradients $\tanh v_1$ and $\tanh v_2$ transform under T into lines with gradients $\tanh v_1'$ and $\tanh v_2'$ respectively. Find the relation satisfied by v_1 and v_2 that is the necessary and sufficient for ℓ_1 and ℓ_2 to intersect at the same angle as their transforms. In the case when ℓ_1 and ℓ_2 meet at the origin, illustrate in a diagram the relation between ℓ_1, ℓ_2 and their transforms.

None

Question (1992 STEP III Q1) **(i)** Given that

$$\mathrm{f}(x) = \ln(1 + \mathrm{e}^x),$$

prove that $\ln[\mathrm{f}'(x)] = x - \mathrm{f}(x)$ and that $\mathrm{f}''(x) = \mathrm{f}'(x) - [\mathrm{f}'(x)]^2$. Hence, or otherwise, expand $\mathrm{f}(x)$ as a series in powers of x up to the term in x^4.

(ii) Given that

$$\mathrm{g}(x) = \frac{1}{\sinh x \cosh 2x},$$

explain why $\mathrm{g}(x)$ can not be expanded as a series of non-negative powers of x but that $x\mathrm{g}(x)$ can be so expanded. Explain also why this latter expansion will consist of even powers of x only. Expand $x\mathrm{g}(x)$ as a series as far as the term in x^4.

(i)

$$\begin{aligned}
f(x) &= \ln(1 + e^x) \\
f'(x) &= \frac{1}{1+e^x} \cdot e^x \\
&= \frac{e^x}{1+e^x} \\
\Rightarrow \quad \ln[f'(x)] &= x - \ln(1 + e^x) \\
&= x - f(x)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \frac{f''(x)}{f'(x)} &= 1 - f'(x) \\
\Rightarrow \quad f''(x) &= f'(x) - [f'(x)]^2 \\
f'''(x) &= f''(x) - 2f'(x)f''(x) \\
f^{(4)}(x) &= f'''(x) - 2[f''(x)]^2 - 2f'(x)f'''(x)
\end{aligned}$$

$$\begin{aligned}
f(0) &= \ln 2 \\
f'(0) &= \tfrac{1}{2} \\
f''(0) &= \tfrac{1}{2} - \tfrac{1}{4} \\
&= \tfrac{1}{4} \\
f'''(0) &= \tfrac{1}{4} - 2\tfrac{1}{2}\tfrac{1}{4} \\
&= 0 \\
f^{(4)}(0) &= -2 \cdot \tfrac{1}{16} \\
&= -\frac{1}{8}
\end{aligned}$$

Therefore $f(x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{8 \cdot 4!}x^4 + O(x^5)$

(ii) As $x \to 0$, $g(x) \to \infty$ therefore there can be no power series about 0. But as $x \to 0, xg(x) \not\to \infty$ as $\frac{x}{\sinh x}$ is well behaved.

We can also notice that $xg(x)$ is an even function, since $\cosh x$ is even and $\frac{x}{\sinh x}$ is even, therefore the power series will consist of even powers of x

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sinh x\cosh 2x} &= \lim_{x\to 0}\frac{x}{\sinh x}\cdot\lim_{x\to 0}\frac{1}{\cosh 2x}\\ &= 1\end{aligned}$$

Notice that

$$\begin{aligned}\frac{x}{\sinh x\cosh 2x} &= \frac{4x}{(e^x-e^{-x})(e^{2x}+e^{-2x})}\\ &= \frac{4x}{(2x+\frac{x^3}{3}+\cdots)(2+4x^2+\frac{4}{3}x^4+\cdots)}\\ &= \frac{1}{1+\frac{x^2}{6}+\frac{x^4}{5!}+\cdots}\frac{1}{1+2x^2+\frac{2}{3}x^4+\cdots}\\ &= \left(1-(\frac{x^2}{6}+\frac{x^4}{5!})+(\frac{x^2}{6})^2+O(x^6)\right)\left(1-(2x^2+\frac{2}{3}x^4)+(2x^2)^2+O(x^6)\right)\\ &= \left(1-\frac{1}{6}x^2+\frac{7}{360}x^4+O(x^6)\right)\left(1-2x^2+\frac{10}{3}x^4+O(x^6)\right)\\ &= 1-\frac{13}{6}x^2+\frac{1327}{360}x^4+O(x^6)\end{aligned}$$

Question (1993 STEP III Q7)

The real numbers x and y satisfy the simultaneous equations

$$\sinh(2x) = \cosh y \qquad \text{and} \qquad \sinh(2y) = 2\cosh x.$$

Show that $\sinh^2 y$ is a root of the equation

$$4t^3+4t^2-4t-1=0$$

and demonstrate that this gives at most one valid solution for y. Show that the relevant value of t lies between 0.7 and 0.8, and use an iterative process to find t to 6 decimal places. Find y and hence find x, checking your answers and stating the final answers to four decimal places.

Let $t = \sinh^2 y$, then

$$\sinh(2x) = \cosh y \qquad (1)$$
$$\sinh(2y) = 2\cosh x \qquad (2)$$

$$\begin{aligned}\cosh(2x) &= 2\cosh^2 x - 1\\ (2): \quad &= \frac{1}{2}\sinh^2(2y)-1\end{aligned}$$

$$\begin{aligned}
1 &= \left(\frac{1}{2}\sinh^2(2y) - 1\right)^2 - \cosh^2 y \\
&= \frac{1}{4}\sinh^4(2y) - \sinh^2(2y) + 1 - \cosh^2 y \\
\Rightarrow \qquad 0 &= \frac{1}{4}(\cosh^2(2y) - 1)^2 - (\cosh^2(2y) - 1) - \cosh^2 y \\
&= \frac{1}{4}\left(\left(1 + 2\sinh^2 y\right)^2 - 1\right)^2 - \left(\left(1 + 2\sinh^2 y\right)^2 - 1\right) - (1 + \sinh^2 y) \\
&= \frac{1}{4}\left(1 + 4t + 4t^2 - 1\right)^2 - \left(1 + 4t + 4t^2 - 1\right) - (1 + t) \\
&= \frac{1}{4}(4t + 4t^2)^2 - (4t + 4t^2) - 1 - t \\
&= 4(t + t^2)^2 - 4t^2 - 5t - 1 \\
&= 4t^4 + 8t^3 + 4t^2 - 4t^2 - 5t - 1 \\
&= 4t^4 + 8t^3 - 5t - 1 \\
&= (t + 1)(4t^3 + 4t^2 - 4t - 1)
\end{aligned}$$

Since $\sinh^2 y$ is positive, we must be a root of the second cubic.

Let $f(t) = 4t^3 + 4t^2 - 4t - 1$, then $f(0) = -1$ and $f'(t) = 12t^2 + 8t - 4 = 4(t+1)(3t-1)$, so we have turning points at -1 and $\frac{1}{3}$. Since $f(-1) = 3 > 0$ and $f(0) < 0$ we must have exactly one root larger than zero. Therefore there is a unique root.

$f(0.7) = -0.468 < 0$ $f(0.8) = 0.408 > 0$

since f is continuous and changes sign, the root must fall in the interval $(0.7, 0.8)$.

Let $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$, and $t_0 = 0.75$, then

$$\begin{aligned}
t_0 &= 0.75 \\
t_1 &= 0.7571428571 \\
t_2 &= 0.7570684728 \\
t_3 &= 0.7570684647
\end{aligned}$$

So $t \approx 0.757068$, $\sinh y \approx 0.870097$, $y \approx 0.786474$, $x \approx 0.546965$

Question (1996 STEP III Q1)

Define $\cosh x$ and $\sinh x$ in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \tfrac{1}{4}\cosh 4x + \tfrac{3}{4}.$$

Find $a_0, a_1, \ldots, a_n$ in terms of n such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx.$$

Hence, or otherwise, find expressions for $\cosh^{2m} x - \sinh^{2m} x$ and $\cosh^{2m} x + \sinh^{2m} x$, in terms of $\cosh kx$, where $k = 0, \ldots, 2m$.

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\begin{aligned}
\cosh^4 x - \sinh^4 x &= (\cosh^2 x - \sinh^2 x)(\cosh^2 x + \sinh^2 x) \\
&= \left(\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\right)(\cosh^2 x + \sinh^2 x) \\
&= (\cosh^2 x + \sinh^2 x) \\
&= \left(\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) + \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\right) \\
&= \frac{1}{4}\left(2e^{2x} + 2e^{-2x}\right) \\
&= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) \\
&= \cosh 2x
\end{aligned}$$

$$\begin{aligned}
\cosh^4 x + \sinh^4 x &= \frac{1}{2^4}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right) + \frac{1}{2^4}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right) \\
&= \frac{1}{8}(e^{4x} + e^{-4x}) + \frac{3}{4} \\
&= \frac{1}{4}\cosh 4x + \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\cosh^n x &= \frac{1}{2^n}\left(e^x + e^{-x}\right)^n \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{kx} e^{-(n-k)x} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{2kx-nx} \\
&= \frac{1}{2^n}\left(\binom{n}{n}\left(e^{nx} + e^{-nx}\right) + \binom{n}{n-1}\left(e^{(n-2)x} + e^{-(n-2)x}\right) + \cdots + \binom{n}{n-k}\left(e^{(n-2k)x} + e^{-(n-2k)x}\right)\right. \\
&= \frac{1}{2^{n-1}}\cosh nx + \frac{1}{2^{n-1}}\binom{n}{n-1}\cosh(n-2)x + \cdots + \frac{1}{2^{n-1}}\binom{n}{n-k}\cosh(n-2k)x + \cdots
\end{aligned}$$

ie

$$\begin{aligned}
\cosh^{2m} x &= \frac{1}{2^{2m-1}}\cosh 2mx + \frac{2m}{2^{2m-1}}\cosh(2(m-1)x) + \cdots + \frac{1}{2^{2m-1}}\binom{2m}{k}\cosh(2(m-k)x) + \cdot \\
\sinh^{2m} x &= \frac{1}{2^{2m-1}}\cosh 2mx - \frac{2m}{2^{2m-1}}\cosh(2(m-1)x) + \cdots + (-1)^k\frac{1}{2^{2m-1}}\binom{2m}{k}\cosh(2(m-k) \\
\cosh^{2m} x - \sinh^{2m} x &= \frac{m}{2^{2m-3}}\cosh(2(m-1)x) + \cdots + \frac{1}{2^{2m-2}}\binom{2m}{2k+1}\cosh(2(m-2k-1)x) + \cdots \\
\cosh^{2m} x + \sinh^{2m} x &= \frac{1}{2^{2m-2}}\cosh(2mx) + \cdots + \frac{1}{2^{2m-2}}\binom{2m}{2k}\cosh(2(m-2k)x) + \cdots
\end{aligned}$$

Question (2006 STEP III Q7) **(i)** Solve the equation $u^2 + 2u \sinh x - 1 = 0$ giving u in terms of x. Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = f(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

Question (2007 STEP III Q5)
Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth \theta = x$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cosh\theta}{r} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\cosh 2\theta}{r^2},$$

and find an expression in terms of r and θ for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Find, with proof, a similar formula for $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of r and θ.

$$\begin{aligned}
y &= \ln(x^2 - 1) \\
r &= \sqrt{x^2 - 1} \\
\coth\theta &= x \\
r &= \sqrt{\coth^2\theta - 1} = \sqrt{\text{cosech}^2\theta} = \text{cosech}\theta \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x}{x^2 - 1} \\
&= \frac{2\coth\theta}{r^2} \\
&= \frac{2\cosh\theta}{\sinh\theta \cdot r \cdot \text{cosech}\theta} \\
&= \frac{2\cosh\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2(x^2 - 1) - 4x^2}{(x^2 - 1)^2} \\
&= \frac{-2(1 + x^2)}{r^2\text{cosech}^2 r} \\
&= -\frac{2(1 + \coth^2\theta)\sinh^2\theta}{r^2}
\end{aligned}$$

$$= -\frac{2(\sinh^2\theta + \cosh^2\theta)}{r^2}$$
$$= -\frac{2\cosh 2\theta}{r^2}$$

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \frac{-4x(x^2-1)^2 - (-2x^2-2)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} \\ &= \frac{-4x(x^2-1) + 8x(x^2+1)}{(x^2-1)^3} \\ &= \frac{4x^3+12x}{(x^2-1)^3} \\ &= \frac{\sinh^3\theta(4\coth^3\theta + 12\coth\theta)}{r^3} \\ &= \frac{4\cosh^3\theta + 12\cosh\theta\sinh^2\theta}{r^3} \\ &= \frac{4\cosh 3\theta}{r^3}\end{aligned}$$

Claim: $\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = (-1)^{n+1}\frac{2(n-1)!\cosh n\theta}{r^n}$ Proof: By induction. Base cases already proven

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}x} &= \frac{x}{\sqrt{x^2-1}} = \frac{\coth\theta}{\text{cosech}\theta} = \cosh\theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}x} &= -\sinh^2\theta\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} &= (-1)^{n+1}(n-1)!\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\cosh n\theta}{r^n}\right) \\ &= (-1)^{n+1}\frac{2n\sinh n\theta\cdot r^n\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x} - 2\cosh n\theta\cdot nr^{n-1}\frac{\mathrm{d}r}{\mathrm{d}x}}{r^{2n}} \\ &= (-1)^{n+2}\frac{2n(\cosh n\theta\cosh\theta + r\sinh n\theta\sinh^2\theta)}{r^{n+1}} \\ &= (-1)^{n+2}n!\frac{2\cosh(n+1)\theta}{r^{n+1}}\end{aligned}$$

We can think of this as $\ln(x^2-1) = \ln(x+1) + \ln(x-1)$ and also note $x \pm 1 = \coth\theta \pm 1 = \frac{\cosh\theta \pm \sinh\theta}{\sinh\theta} = \frac{e^{\pm\theta}}{\sinh\theta}$

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\ln(x^2-1) &= (n-1)!(-1)^{n-1}\left(\frac{1}{(x+1)^n} + \frac{1}{(x-1)^n}\right) \\ &= (-1)^{n-1}(n-1)!\left(\frac{\sinh^n\theta}{e^{n\theta}} + \frac{\sinh^n\theta}{e^{-n\theta}}\right) \\ &= (-1)^{n-1}(n-1)!2\cosh n\theta\cdot\sinh^n\theta \\ &= (-1)^{n-1}(n-1)!\frac{2\cosh n\theta}{r^n}\end{aligned}$$

Question (2014 STEP III Q6)

Starting from the result that

$$\dot{h}(t) > 0 \text{ for } 0 < t < x \Longrightarrow \int_0^x \dot{h}(t)t > 0\,,$$

show that, if $\dot{f}''(t) > 0$ for $0 < t < x_0$ and $\dot{f}(0) = \dot{f}'(0) = 0$, then $\dot{f}(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1\,.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}\,.$$

None

Question (2016 STEP III Q6)

Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = x$ and $y = a\tanh x + b$, where $a > 0$.

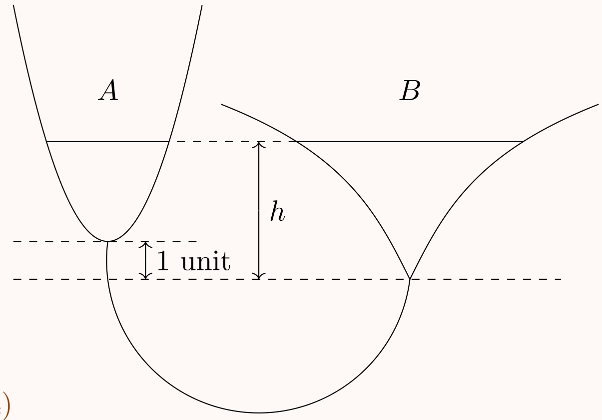
(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm\left(\frac{1}{\sqrt{b^2 - a^2}}\right) - artanh\,\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.



Question (1987 STEP III Q4)

Two funnels A and B have surfaces formed by rotating the curves $y = x^2$ and $y = 2\sinh^{-1} x$ $(x > 0)$ above the y-axis. The bottom of B is one unit lower than the bottom of A and they are connected by a thin rubber tube with a tap in it. The tap is closed and A is filled with water to a depth of 4 units. The tap is then opened. When the water comes to rest, both surfaces are at a height h above the bottom of B, as shown in the diagram. Show that h satisfies the equation

$$h^2 - 3h + \sinh h = 15.$$

The initial volume of water in A is:

$$\begin{aligned}\pi \int_0^4 x^2 \, \mathrm{d}y &= \pi \int_0^4 y \mathrm{d}y \\ &= \pi [\frac{y^2}{2}]_0^4 \\ &= 8\pi\end{aligned}$$

We assume that no water is in the tube as it is 'thin'.
Therefore we must have:

$$\begin{aligned}8\pi &= \pi \int_0^{h-1} x^2 \mathrm{d}y + \pi \int_0^h x^2 \mathrm{d}y \\ &= \pi \int_0^{h-1} y \mathrm{d}y + \pi \int_0^h \left(\sinh \frac{x}{2}\right)^2 \mathrm{d}y \\ &= \pi \left[\frac{y^2}{2}\right]_0^{h-1} + \pi \int_0^h \frac{-1 + \cosh y}{2} \mathrm{d}y \\ &= \pi \frac{(h-1)^2}{2} + \pi \left[-\frac{y}{2} + \frac{\sinh y}{2}\right]_0^h \\ &= \pi \frac{(h-1)^2}{2} - \pi \frac{h}{2} + \pi \frac{\sinh h}{2} \\ \Rightarrow \qquad 0 &= h^2 - 2h + 1 - h + \sinh h - 16 \\ &= h^2 - 3h + \sinh h - 15 \\ \Rightarrow \qquad 15 &= h^2 - 3h + \sinh h\end{aligned}$$