Question (1989 STEP I Q2)

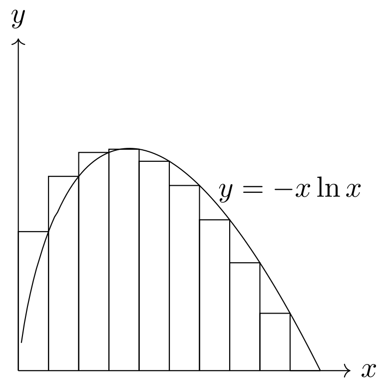
For $x > 0$ find $\int x \ln x \, dx$. By approximating the area corresponding to $\int_0^1 x \ln(1/x) \, dx$ by n rectangles of equal width and with their top right-hand vertices on the curve $y = x \ln(1/x)$, show that, as $n \to \infty$,

$$\frac{1}{2}\left(1+\frac{1}{n}\right)\ln n - \frac{1}{n^2}\left[\ln\left(\frac{n!}{0!}\right) + \ln\left(\frac{n!}{1!}\right) + \ln\left(\frac{n!}{2!}\right) + \cdots + \ln\left(\frac{n!}{(n-1)!}\right)\right] \to \frac{1}{4}.$$

[You may assume that $x \ln x \to 0$ as $x \to 0$.]

Integrating by parts we obtain:

$$\begin{aligned} \int x \ln x \, dx &= [\frac{1}{2}x^2 \ln x] - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \end{aligned}$$



We should have:

$$\begin{aligned} \int_0^1 x \ln \frac{1}{x} dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\frac{i}{n} \ln\left(\frac{n}{i}\right) \\ \left[-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right]_0^1 &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\frac{i}{n} \ln\left(\frac{n}{i}\right) \\ \frac{1}{4} &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} (i \ln n - i \ln i) \\ &= \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{n(n+1)}{2} \ln n - \sum_{i=1}^{n} i \ln i \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2}(1+\frac{1}{n}) \ln n - \frac{1}{n^2} \sum_{i=1}^{n} i \ln i \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2}(1+\frac{1}{n}) \ln n - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{i} \ln i \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2}(1+\frac{1}{n}) \ln n - \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{i=0}^{k} \ln(n-i) \right) \end{aligned}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}(1+\frac{1}{n})\ln n - \frac{1}{n^2}\sum_{k=0}^{n-1}\ln\frac{n!}{(n-k)!}\right)$$

Question (1989 STEP III Q9)

Obtain the sum to infinity of each of the following series.

(i) $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{r}{2^{r-1}}+\cdots;$

(ii) $1+\frac{1}{2}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2^2}+\cdots+\frac{1}{r}\times\frac{1}{2^{r-1}}+\cdots;$

(iii) $\frac{1\times 3}{2!}\times\frac{1}{3}+\frac{1\times 3\times 5}{3!}\frac{1}{3^2}+\cdots+\frac{1\times 3\times\cdots\times(2k-1)}{k!}\times\frac{1}{3^{k-1}}+\cdots.$

[Questions of convergence need not be considered.]

(i)

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$$

$$\underbrace{\Rightarrow}_{\frac{d}{dx}} \quad \frac{1}{(1-x)^2} = \sum_{r=0}^{\infty} rx^{r-1}$$

$$\underbrace{\Rightarrow}_{x=\frac{1}{2}} \quad 4 = \sum_{r=0}^{\infty}\frac{r}{2^{r-1}}$$

(ii)

$$\frac{1}{1-x} = \sum_{r=1}^{\infty} x^{r-1}$$

$$\underbrace{\Rightarrow}_{\int} \quad -\ln(1-x) = \sum_{r=1}^{\infty}\frac{1}{r}x^r$$

$$\underbrace{\Rightarrow}_{x=\frac{1}{2}} \quad \ln 2 = \sum_{r=1}^{\infty}\frac{1}{r}\times\frac{1}{2^r}$$

$$\Rightarrow \quad 2\ln 2 = \sum_{r=1}^{\infty}\frac{1}{r}\times\frac{1}{2^{r-1}}$$

(iii)

$$(1-x)^{-1/2} = 1+\frac{(-\frac{1}{2})}{1!}(-x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x)^2+\cdots$$

$$= \sum_{r=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2r-1)}{2^r r!} x^r$$

$$\underbrace{\Rightarrow}_{x=\frac{2}{3}} \qquad \sqrt{3} = \sum_{r=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2r-1)}{r!} \frac{1}{3^r}$$

$$= 1 + \frac{1}{1!}\frac{2}{3} + \frac{1}{3}\sum_{r=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2r-1)}{r!} \frac{1}{3^{r-1}}$$

$$\Rightarrow \qquad 3\sqrt{3} - 5 = \sum_{r=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2r-1)}{r!} \frac{1}{3^{r-1}}$$

Question (1993 STEP III Q4)

Sum the following infinite series.

(i)

$$1 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{5}\left(\frac{1}{2}\right)^4 + \cdots + \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n} + \cdots$$

.

(ii)

$$2 - x - x^3 + 2x^4 - \cdots + 2x^{4k} - x^{4k+1} - x^{4k+3} + \cdots$$

where $|x| < 1$.

(iii)

$$\sum_{r=2}^{\infty} \frac{r\,2^{r-2}}{3^{r-1}}$$

.

(iv)

$$\sum_{r=2}^{\infty} \frac{2}{r(r^2-1)}$$

.

(i)

$$\sum_{i=0}^{\infty} x^{2i+1} = \frac{x}{1-x^2}$$

$$\Rightarrow \qquad = \frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)$$

$$\underbrace{\Rightarrow}_{\int} \qquad \sum_{i=0}^{\infty} \frac{1}{2i+1} x^{2i+2} = \frac{1}{2}\left(-\ln(1-x) - \ln(1+x)\right)$$

$$\underbrace{\Rightarrow}_{x=1/2} \qquad \sum_{i=0}^{\infty} \frac{1}{2i+1}\left(\frac{1}{2}\right)^{2i+2} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{3}{2}$$

$$= -\frac{1}{2}\ln\frac{3}{4}$$

$$\frac{1}{4}\sum_{i=0}^{\infty} \frac{1}{2i+1}\left(\frac{1}{2}\right)^{2i} = \frac{1}{2}\ln\frac{4}{3}$$

$$\Rightarrow \qquad S = 2\ln\frac{4}{3}$$

(ii)

$$\sum_{k=0}^{\infty}\left(2x^{4k} - x^{4k+1} - x^{4k+3}\right) = \sum_{k=0}^{\infty}\left(2 - x^1 - x^3\right)x^{4k}$$

$$= \frac{2 - x - x^3}{1 - x^4}$$

$$= \frac{(1-x)(2+x+x^2)}{(1-x)(1+x+x^2+x^3)}$$

$$= \frac{2+x+x^2}{1+x+x^2+x^3}$$

(iii)

$$\frac{1}{(1-x)^2} = \sum_{r=0}^{\infty} r x^{r-1}$$

$$\Rightarrow \qquad 9 = \sum_{r=1}^{\infty} r\left(\frac{2}{3}\right)^{r-1}$$

$$\Rightarrow \qquad \sum_{r=2}^{\infty} r\left(\frac{2^{r-2}}{3^{r-1}}\right) = \frac{1}{2}(9-1)$$

$$= 4$$

(iv)

$$\frac{2}{r(r^2-1)} = \frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}$$

$$\Rightarrow \qquad \sum_{r=2}^{\infty}\left(\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}\right) = \sum_{r=2}^{\infty}\left(\frac{1}{r-1} - \frac{1}{r} - \frac{1}{r} + \frac{1}{r+1}\right)$$

$$= \sum_{r=2}^{\infty}\left(\frac{1}{r-1} - \frac{1}{r}\right) - \sum_{r=2}^{\infty}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

Question (1997 STEP III Q7)

For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots ;$$
$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots .$$

(i) Evaluate b_n.

(ii) Show that $0 < a_n < 1/n$.

(iii) Deduce that $a_n = n!\mathrm{e} - [n!\mathrm{e}]$ (where $[x]$ is the integer part of x).

(iv) Hence show that e is irrational.

Question (1998 STEP II Q3)

Show that the sum S_N of the first N terms of the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{\cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \cdots + \frac{2n-1}{n(n+1)(n+2)} + \cdots$$

is

$$\frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right).$$

What is the limit of S_N as $N \to \infty$? The numbers a_n are such that

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for a_n/a_1 and hence, or otherwise, evaluate $\sum_{n=1}^{\infty} a_n$ when $a_1 = \frac{2}{9}$.

First notice by partial fractions:

$$\begin{aligned}\frac{2n-1}{n(n+1)(n+2)} &= \frac{-1/2}{n} + \frac{3}{n+1} + \frac{-5/2}{n+2} \\ &= \frac{-1}{2n} + \frac{3}{n+1} - \frac{5}{2(n+2)}\end{aligned}$$

And therefore:

$$\begin{aligned}\sum_{n=1}^{N} \frac{2n-1}{n(n+1)(n+2)} &= -\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} + 3\sum_{n=1}^{N}\frac{1}{n+1} - \frac{5}{2}\sum_{n=1}^{N}\frac{1}{n+2} \\ &= -\frac{1}{2} - \frac{1}{4} + \frac{3}{2} + \sum_{n=3}^{N}(3 - \frac{1}{2} - \frac{5}{2})\frac{1}{n} + \frac{3}{N+1} - \frac{5}{2(N+1)} - \frac{5}{2(N+2)} \\ &= \frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right)\end{aligned}$$

As $N \to \infty, S_N \to \frac{3}{4}$.

$$\begin{aligned}
\frac{a_n}{a_{n-1}} &= \frac{(n-1)(2n-1)}{(n+2)(2n-3)} \\
\Rightarrow \quad \frac{a_n}{a_1} &= \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \\
&= \frac{(n-1)(2n-1)}{(n+2)(2n-3)} \cdot \frac{(n-2)(2n-3)}{(n+1)(2n-5)} \cdots \frac{(1)(3)}{(4)(1)} \\
&= \frac{(2n-1)3 \cdot 2 \cdot 1}{(n+2)(n+1)n} \\
&= \frac{6(2n-1)}{n(n+1)(n+2)}
\end{aligned}$$

Therefore $a_n = \frac{4}{3}\frac{2n-1}{n(n+1)(n+2)}$ and so our sequence is $\frac{4}{3}$ the earlier sum, ie 1

Question (1999 STEP III Q3)

Justify, by means of a sketch, the formula

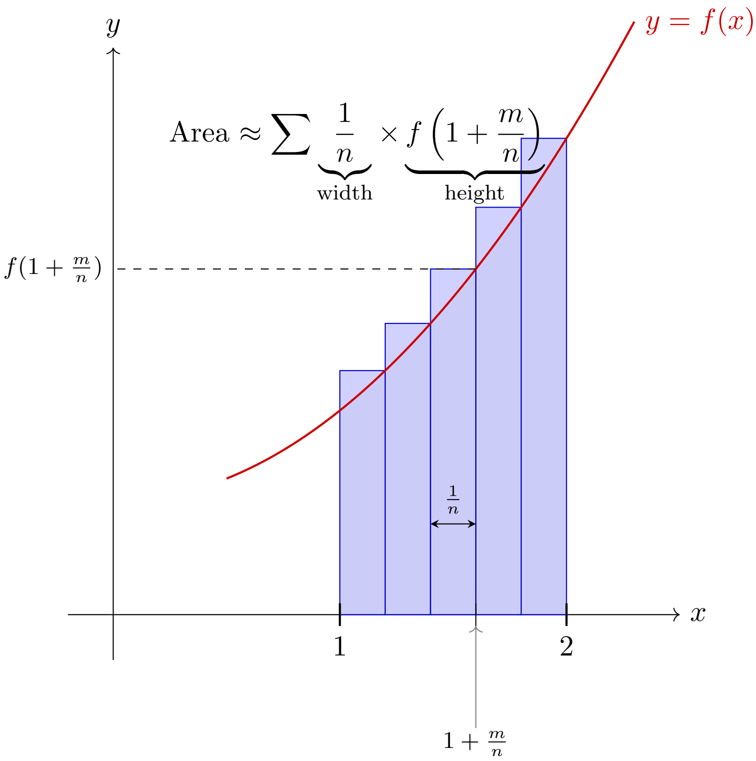
$$\lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{m=1}^{n} f(1 + m/n) \right\} = \int_1^2 f(x)\, \mathrm{d}x\,.$$

Show that

$$\lim_{n\to\infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right\} = \ln 2\,.$$

Evaluate

$$\lim_{n\to\infty} \left\{ \frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2} \right\}.$$



$$
\begin{aligned}
V &= \lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}\\
&= \lim_{n\to\infty}\left\{\sum_{m=1}^{n}\frac{1}{n+m}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{1+\frac{m}{n}}\right\}\\
&= \int_1^2 \frac{1}{x}\,\mathrm{d}x\\
&= [\ln x]_1^2 = \ln 2
\end{aligned}
$$

$$
\begin{aligned}
V &= \lim_{n\to\infty}\left\{\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+n^2}\right\}\\
&= \lim_{n\to\infty}\left\{\sum_{m=1}^{n}\frac{n}{n^2+m^2}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{1+\left(\frac{m}{n}\right)^2}\right\}\\
&= \int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x\\
&= \left[\tan^{-1}x\right]_0^1\\
&= \frac{\pi}{4}
\end{aligned}
$$

Question (1999 STEP III Q5)

The sequence u_0, u_1, u_2, ... is defined by

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad \text{for} \quad n \geq 1\,.$$

Prove that

$$u_{n+2}^2 + u_{n-1}^2 = 2(u_{n+1}^2 + u_n^2)\,.$$

Using induction, or otherwise, prove the following result:

$$u_{2n} = u_n^2 + u_{n-1}^2 \quad \text{and} \quad u_{2n+1} = u_{n+1}^2 - u_{n-1}^2$$

for any positive integer n.

Claim: $u_{n+2}^2 + u_{n-1}^2 = 2(u_{n+1}^2 + u_n^2)$

Proof: (By Induction).

(Base Case): $n = 1$

$$\begin{aligned} LHS &= u_{n+2}^2 + u_{n-1}^2 \\ &= u_3^2 + u_0^2 \\ &= 3^2 + 1^2 = 10 \\ RHS &= 2(u_{n+1}^2 + u_n^2) \\ &= 2(2^2 + 1^2) \\ &= 10 \end{aligned}$$

Therefore the base case is true.

(Inductive hypothesis) Suppose $u_{n+2}^2 + u_{n-1}^2 = 2(u_{n+1}^2 + u_n^2)$ is true for some $n = k$, ie $u_{k+2}^2 + u_{k-1}^2 = 2(u_{k+1}^2 + u_k^2)$, the consider $n = k + 1$

$$\begin{aligned} LHS &= u_{k+1+2}^2 + u_{k+1-1}^2 \\ &= (u_{k+1} + u_{k+2})^2 + u_k^2 \\ &= u_{k+2}^2 + u_{k+1}^2 + u_k^2 + 2u_{k+1}u_{k+2} \\ &= u_{k+2}^2 + u_{k+1}^2 + u_k^2 + 2u_{k+1}(u_{k+1} + u_k) \\ &= u_{k+2}^2 + u_{k+1}^2 + u_k^2 + 2u_{k+1}^2 + 2u_{k+1}u_k \\ &= u_{k+1}^2 + 2u_{k+1}^2 + u_{k+1}^2 + u_k^2 + 2u_{k+1}u_k \\ &= u_{k+2}^2 + 2u_{k+1}^2 + (u_{k+1} + u_k)^2 \\ &= u_{k+2}^2 + 2u_{k+1}^2 + u_{k+2}^2 \\ &= 2(u_{k+2}^2 + u_{k+1}^2) \\ &= RHS \end{aligned}$$

Therefore it is true for $n = k + 1$.

Therefore by the principle of mathematical induction it is true for all $n \geq 1$

Claim: $u_{2n} = u_n^2 + u_{n-1}^2 \quad \text{and} \quad u_{2n+1} = u_{n+1}^2 - u_{n-1}^2$

Proof: Notice that $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, in particular

$$\begin{pmatrix} u_n & u_{n-1} \\ u_{n-1} & u_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

$$\Rightarrow \qquad \begin{pmatrix} u_{2n} & u_{2n-1} \\ u_{2n-1} & u_{2n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2n}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n}$$
$$= \begin{pmatrix} u_n & u_{n-1} \\ u_{n-1} & u_{n-2} \end{pmatrix} \begin{pmatrix} u_n & u_{n-1} \\ u_{n-1} & u_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} u_n^2 + u_{n-1}^2 & u_{n-1}(u_n + u_{n-2}) \\ u_{n-1}(u_n + u_{n-2}) & u_{n-1}^2 + u_{n-2}^2 \end{pmatrix}$$

Therefore $u_{2n} = u_n^2 + u_{n-1}^2$ and $u_{2n+1} = u_n(u_{n+1} + u_{n-1}) = (u_{n+1} - u_{n-1})(u_{n+1} - u_{n-1}) = u_{n+1}^2 - u_{n-1}^2$

Question (2000 STEP III Q7)

Given that

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots ,$$

use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < \mathrm{e}$$

for any positive integer n.

The product $\mathrm{P}(n)$ is defined, for any positive integer n, by

$$\mathrm{P}(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \ldots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \left(a_1 \cdot a_2 \cdot \ldots \cdot a_n\right)^{\frac{1}{n}},$$

to show that $\mathrm{P}(n) < \mathrm{e}$ for all n .

Explain briefly why $\mathrm{P}(n)$ tends to a limit as $n \to \infty$. Show that this limit, L, satisfies $2 < L \leq \mathrm{e}$.

None

Question (2003 STEP II Q7)

Show that, if $n > 0$, then

$$\int_{e^{1/n}}^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x = \frac{2}{n^2\mathrm{e}}\,.$$

You may assume that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$.

Explain why, if $1 < a < b$, then

$$\int_b^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x < \int_a^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x\,.$$

Deduce that

$$\sum_{n=1}^{N} \frac{1}{n^2} < \frac{\mathrm{e}}{2}\int_{\mathrm{e}^{1/N}}^{\infty} \left(\frac{1-x^{-N}}{x^2-x}\right)\ln x\,\mathrm{d}x\,,$$

where N is any integer greater than 1.

None

Question (2003 STEP III Q6)

Show that

$$2\sin\frac{1}{2}\theta\,\cos r\theta = \sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta\,.$$

Hence, or otherwise, find all solutions of the equation

$$\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0\,,$$

where a and b are positive integers with $a < b-1$.

$$\begin{aligned}\sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta &= \sin r\theta\cos\tfrac{1}{2}\theta + \cos r\theta\sin\tfrac{1}{2}\theta - \left(\sin r\theta\cos\tfrac{1}{2}\theta - \cos r\theta\sin\tfrac{1}{2}\theta\right)\\ &= 2\cos r\theta\sin\tfrac{1}{2}\theta\end{aligned}$$

$$\begin{aligned}S &= \cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta\\ 2\sin\tfrac{1}{2}\theta S &= \sum_{r=a}^{b-1} 2\sin\tfrac{1}{2}\theta\cos r\theta\\ &= \sum_{r=a}^{b-1}\left(\sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta\right)\\ &= \sin\left(b-\frac{1}{2}\right)\theta - \sin\left(a-\frac{1}{2}\right)\theta\\ \Rightarrow \qquad \sin\left(b-\frac{1}{2}\right)\theta &= \sin\left(a-\frac{1}{2}\right)\theta\end{aligned}$$

Case 1: $A = B + 2n\pi$

$$\left(b-\frac{1}{2}\right)\theta = \left(a-\frac{1}{2}\right)\theta + 2n\pi$$

$$\Rightarrow \qquad (b-a)\theta = 2n\pi$$

$$\Rightarrow \qquad \theta = \frac{2n\pi}{b-a}$$

Case 2: $A = (2n+1)\pi - B$

$$\left(b - \frac{1}{2}\right)\theta = (2n+1)\pi - \left(a - \frac{1}{2}\right)\theta$$

$$\Rightarrow \qquad (b+a-1)\theta = (2n+1)\pi$$

$$\Rightarrow \qquad \theta = \frac{2n\pi}{b+a-1}$$

Question (2004 STEP I Q8)

A sequence t_0, t_1, t_2, ... is said to be $\mathfrak{sl}\, strictlyincreasingif\mathrm{t}_{n+1} > t_n$ for all $n \geq 0$.

(i) The terms of the sequence x_0, x_1, x_2, ... satisfy

$$x_{n+1} = \frac{x_n^2 + 6}{5}$$

for $n \geq 0$. Prove that if $x_0 > 3$ then the sequence is strictly increasing.

(ii) The terms of the sequence y_0, y_1, y_2, ... satisfy

$$y_{n+1} = 5 - \frac{6}{y_n}$$

for $n \geq 0$. Prove that if $2 < y_0 < 3$ then the sequence is strictly increasing but that $y_n < 3$ for all n.

Question (2004 STEP III Q3)

Given that $f''(x) > 0$ when $a \leq x \leq b$, explain with the aid of a sketch why

$$(b-a)\,f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,\mathrm{d}x < (b-a)\,\frac{f(a)+f(b)}{2}\,.$$

By choosing suitable a, b and $f(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where n is an integer greater than 1. Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}\,.$$

None

Question (2004 STEP III Q6)

Given a sequence w_0, w_1, w_2, $\ldots$, the sequence F_1, F_2, $\ldots$ is defined by

$$F_n = w_n^2 + w_{n-1}^2 - 4w_nw_{n-1}\,.$$

Show that $F_n - F_{n-1} = (w_n - w_{n-2})\,(w_n + w_{n-2} - 4w_{n-1})$ for $n \ge 2$.

(i) The sequence u_0, u_1, u_2, $\ldots$ has $u_0 = 1$, and $u_1 = 2$ and satisfies

$$u_n = 4u_{n-1} - u_{n-2} \quad (n \ge 2)\,.$$

Prove that $u_n^2 + u_{n-1}^2 = 4u_nu_{n-1} - 3$ for $n \ge 1$.

(ii) A sequence v_0, v_1, v_2, $\ldots$ has $v_0 = 1$ and satisfies

$$v_n^2 + v_{n-1}^2 = 4v_nv_{n-1} - 3 \quad (n \ge 1). \qquad (*)$$

(a) Find v_1 and prove that, for each $n \ge 2$, either $v_n = 4v_{n-1} - v_{n-2}$ or $v_n = v_{n-2}$.

(b) Show that the sequence, with period 2, defined by

$$v_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 2 & \text{for } n \text{ odd} \end{cases}$$

satisfies $(*)$. (c) Find a sequence v_n with period 4 which has $v_0 = 1$, and satisfies $(*)$.

Question (2005 STEP III Q4)

The sequence u_n $(n = 1, 2, \ldots)$ satisfies the recurrence relation

$$u_{n+2} = \frac{u_{n+1}}{u_n}(ku_n - u_{n+1})$$

where k is a constant.

If $u_1 = a$ and $u_2 = b$, where a and b are non-zero and $b \ne ka$, prove by induction that

$$u_{2n} = \left(\frac{b}{a}\right)u_{2n-1}$$
$$u_{2n+1} = cu_{2n}$$

for $n \ge 1$, where c is a constant to be found in terms of k, a and b. Hence express u_{2n} and u_{2n-1} in terms of a, b, c and n.

Find conditions on a, b and k in the three cases:

(i) the sequence u_n is geometric;

(ii) u_n has period 2;

(iii) the sequence u_n has period 4.

Question (2006 STEP II Q1)

The sequence of real numbers u_1, u_2, u_3, ... is defined by

$$u_1 = 2\,, \qquad \text{and} \qquad u_{n+1} = k - \frac{36}{u_n} \quad \text{for } n \geq 1, \qquad (*)$$

where k is a constant.

(i) Determine the values of k for which the sequence $(*)$ is: **(a)** constant; **(b)** periodic with period 2; **(c)** periodic with period 4.

(ii) In the case $k = 37$, show that $u_n \geq 2$ for all n. Given that in this case the sequence $(*)$ converges to a limit ℓ, find the value of ℓ.

Question (2008 STEP III Q2)

Let $S_k(n) \equiv \sum_{r=0}^{n} r^k$, where k is a positive integer, so that

$$S_1(n) \equiv \tfrac{1}{2}n(n+1) \text{ and } S_2(n) \equiv \tfrac{1}{6}n(n+1)(2n+1)\,.$$

(i) By considering $\sum_{r=0}^{n} \left[(r+1)^k - r^k\right]$, show that

$$kS_{k-1}(n) = (n+1)^k - (n+1) - \binom{k}{2} S_{k-2}(n) - \binom{k}{3} S_{k-3}(n) - \cdots - \binom{k}{k-1} S_1(n)\,. \qquad (*)$$

Obtain simplified expressions for $S_3(n)$ and $S_4(n)$.

(ii) Explain, using $(*)$, why $S_k(n)$ is a polynomial of degree $k+1$ in n. Show that in this polynomial the constant term is zero and the sum of the coefficients is 1.

(i)

$$\begin{aligned}
(n+1)^k &= \sum_{r=0}^{n} \left[(r+1)^k - r^k\right] \\
&= \sum_{r=0}^{n} \left[\left(\binom{k}{0} r^k + \binom{k}{1} r^{k-1} + \binom{k}{2} r^{k-2} + \cdots + \binom{k}{k} 1\right) - r^k\right] \\
&= \sum_{r=0}^{n} \left(\binom{k}{1} r^{k-1} + \binom{k}{2} r^{k-2} + \cdots + \binom{k}{k} 1\right) \\
&= k\sum_{r=0}^{n} r^{k-1} + \binom{k}{2} \sum_{r=0}^{n} r^{k-2} + \cdots + \binom{k}{k} \sum_{r=0}^{n} 1 \\
&= kS_{k-1}(n) + \binom{k}{2} S_{k-2}(n) + \cdots + \binom{k}{k-1} S_1(n) + (n+1)
\end{aligned}$$

$$\Rightarrow \quad kS_{k-1}(n) = (n+1)^k - (n+1) - \binom{k}{2} S_{k-2}(n) - \cdots - \binom{k}{k-1} S_1(n)$$

$$\begin{aligned}
4S_3(n) &= (n+1)^4 - (n+1) - \binom{4}{2}\frac{n(n+1)(2n+1)}{6} - \binom{4}{3}\frac{n(n+1)}{2} \\
&= (n+1)\left((n+1)^3 - 1 - n(2n+1) - 2n\right) \\
&= (n+1)\left(n^3 + 3n^2 + 3n + 1 - 1 - 2n^2 - 3n\right) \\
&= (n+1)\left(n^3 + n^2\right) \\
&= n^2(n+1)^2 \\
\Rightarrow \qquad S_3(n) &= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

$$\begin{aligned}
5S_4(n) &= (n+1)^5 - (n+1) - \binom{5}{2}\frac{n^2(n+1)^2}{4} - \binom{5}{3}\frac{n(n+1)(2n+1)}{6} - \binom{5}{4}\frac{n(n+1)}{2} \\
&= (n+1)\left((n+1)^4 - 1 - \frac{5n^2(n+1)}{2} - \frac{5n(2n+1)}{3} - \frac{5n}{2}\right) \\
&= \frac{n+1}{6}\left(6(n+1)^4 - 6 - 15n^2(n+1) - 10n(2n+1) - 15n\right) \\
&= \frac{n+1}{6}\left(6n^4 + 24n^3 + 36n^2 + 24n + 6 - 6 - 15n^3 - 15n^2 - 20n^2 - 10n - 15n\right) \\
&= \frac{n+1}{6}\left(6n^4 + 9n^3 + n^2 - n\right) \\
&= \frac{(n+1)n(2n+1)(3n^2+3n-1)}{6} \\
\Rightarrow \qquad S_4(n) &= \frac{(n+1)n(2n+1)(3n^2+3n-1)}{30}
\end{aligned}$$

(ii) Proceeding by induction, since $S_k(n)$ is a polynomial of degree $k+1$ for small k, we can see that

$$(k{+}1)S_k(n) = \underbrace{(n+1)^{k+1}}_{\text{poly deg }=k+1} - \underbrace{(n+1)}_{\text{poly deg=1}} - \underbrace{\binom{k+1}{2}S_{k-1}(n)}_{\text{poly deg}=k} - \underbrace{\cdots}_{\text{polys deg}<k} - \underbrace{\binom{k+1}{k}S_1(n)}_{\text{poly deg=1}}$$

therefore $S_k(n)$ is a polynomial of degree $k+1$ (in fact with leading coefficient $\frac{1}{k+1}$. Since $S_k(0) = \sum_{r=0}^{0} r^k = 0$ there is no constant term, and since $S_k(1) = \sum_{r=0}^{1} r^k = 1$ the sum of the coefficients is 1

Question (2010 STEP III Q7)

Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0\,.$$

Obtain a similar equation relating $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Conjecture and prove a relation between $\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

Obtain the first three non-zero terms of the Maclaurin series for y. Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

State the degree of the polynomial.

Question (2012 STEP I Q7)

A sequence of numbers t_0, t_1, t_2, ... satisfies

$$t_{n+2} = pt_{n+1} + qt_n \qquad (n \geq 0),$$

where p and q are real. Throughout this question, x, y and z are non-zero real numbers.

(i) Show that, if $t_n = x$ for all values of n, then $p + q = 1$ and x can be any (non-zero) real number.

(ii) Show that, if $t_{2n} = x$ and $t_{2n+1} = y$ for all values of n, then $q \pm p = 1$. Deduce that either $x = y$ or $x = -y$, unless p and q take certain values that you should identify.

(iii) Show that, if $t_{3n} = x$, $t_{3n+1} = y$ and $t_{3n+2} = z$ for all values of n, then

$$p^3 + q^3 + 3pq - 1 = 0\,.$$

Deduce that either $p + q = 1$ or $(p-q)^2 + (p+1)^2 + (q+1)^2 = 0$. Hence show that either $x = y = z$ or $x + y + z = 0$.

(i) Suppose $t_n = x$ for all n, then we must have

$$\begin{aligned} && x &= px + qx \\ \Leftrightarrow && 1 &= p + q \end{aligned}$$

and this clearly works for any value of x.

(ii) Suppose $t_{2n} = x, t_{2n+1} = y$ for all n, then

$$\begin{aligned} x &= py + qx \\ y &= px + qy \\ \Rightarrow \quad 0 &= py + (q-1)x \\ 0 &= px + (q-1)y \\ \Rightarrow \quad p &= (q-1)\frac{x}{y} = (q-1)\frac{y}{x} \\ \Rightarrow \quad \frac{y}{x} &= \pm 1 \text{ or } q = 1, p = 0 \\ \Rightarrow \quad y &= \pm x \text{ or } (p,q) = (0,1) \end{aligned}$$

(iii) Suppose $t_{3n} = x$, $t_{3n+1} = y$ and $t_{3n+2} = z$, so

$$\begin{aligned} x &= pz + qy \\ y &= px + qz \\ z &= py + qx \end{aligned}$$

$$\begin{aligned} z &= p(px + qz) + q(pz + qy) \\ &= p^2x + 2pqz + q^2y \\ &= p^2(pz + qy) + 2pqz + q^2(px + qz) \\ &= p^3z + p^2qy + 2pqz + q^2px + q^3z \\ &= (p^3 + q^3 + 2pq)z + pq(py + qx) \\ &= (p^3 + q^3 + 2pq)z + pqz \\ &= (p^3 + q^3 + 3pq)z \\ \Rightarrow \quad 0 &= p^3 + q^3 + 3pq - 1 \\ &= (p + q - 1)(p^2 + q^2 + 1 + p + q - pq) \\ &= \tfrac{1}{2}(p + q - 1)((p - q)^2 + (p + 1)^2 + (q + 1)^2) \end{aligned}$$

Therefore $p + q = 1$ or $(p - q)^2 + (p + 1)^2 + (q + 1)^2 = 0 \Rightarrow p = q = -1$.

If $p + q = 1$, then $z = py + (1 - p)x$ and $x = p(py + (1 - p)x) + (1 - p)y \Rightarrow (1 - p + p^2)x = (1 - p + p^2)y \Rightarrow x = y \Rightarrow x = y = z$.

If $p = q = -1$ then adding all the equations we get $x + y + z = -2(x + y + z) \Rightarrow x + y + z = 0$

Note that what is actually going on here is that solutions must be of the form $t_n = \lambda^n$ so the only way to be constant is for $\lambda = 1$ to be a root, the only way for it to be 2-periodic is for $\lambda = -1$ to be a root, and the only way for it to be 3-periodic is for $\lambda = 1, \omega, \omega^2$ to be the roots (although we see this via the classic $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$ which is because of the real constraint in the question.

Question (2012 STEP II Q8)

The positive numbers α, β and q satisfy $\beta - \alpha > q$. Show that

$$\frac{\alpha^2+\beta^2-q^2}{\alpha\beta} - 2 > 0\,.$$

The sequence u_0, u_1, ... is defined by $u_0 = \alpha$, $u_1 = \beta$ and

$$u_{n+1} = \frac{u_n^2 - q^2}{u_{n-1}} \qquad (n \geq 1),$$

where α, β and q are given positive numbers (and α and β are such that no term in the sequence is zero). Prove that $u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$. Prove also that

$$u_{n+1} - pu_n + u_{n-1} = 0$$

for some number p which you should express in terms of α, β and q. Hence, or otherwise, show that if $\beta > \alpha + q$, the sequence is strictly increasing (that is, $u_{n+1} - u_n > 0$ for all n). Comment on the case $\beta = \alpha + q$.

$$\begin{aligned}
&& \beta - \alpha &> q \\
\Rightarrow && (\beta-\alpha)^2 &> q^2 \\
\Rightarrow && \beta^2 + \alpha^2 - 2\beta\alpha &> q^2 \\
\Rightarrow && \alpha^2 + \beta^2 - q^2 - 2\beta\alpha &> 0 \\
\Rightarrow && \frac{\alpha^2+\beta^2-q^2}{\alpha\beta} - 2 &> 0
\end{aligned}$$

$$\begin{aligned}
u_n(u_n + u_{n+2}) &= u_n \cdot \left(u_n + \frac{u_{n+1}^2 - q^2}{u_n} \right) \\
&= u_n^2 + u_{n+1}^2 - q^2 \\
&= u_n^2 + u_{n+1}^2 - (u_n^2 - u_{n-1}u_{n+1}) \\
&= u_{n+1}^2 + u_{n+1}u_{n-1} \\
&= u_{n+1}(u_{n-1} + u_{n+1})
\end{aligned}$$

$$\begin{aligned}
u_{n+1} - pu_n + u_{n-1} &= -pu_n + \frac{u_n(u_{n-2}+u_n)}{u_{n-1}} \\
&= \frac{u_n(u_n - pu_{n-1} + u_{n-2})}{u_{n-1}}
\end{aligned}$$

Therefore if $u_2 - pu_1 + u_0 = 0$ it is always zero, ie if

$$\begin{aligned}
&& u_2 &= p\beta - \alpha \\
&& u_2 &= \frac{\beta^2 - q^2}{\alpha} \\
\Rightarrow && \frac{\beta^2-q^2}{\alpha} &= p\beta - \alpha \\
\Rightarrow && p &= \frac{\alpha^2+\beta^2-q^2}{\alpha\beta}
\end{aligned}$$

If $\beta > \alpha + q$ we must have that $p > 2$, and so $u_{n+1} - u_n = (p-1)u_n - u_{n-1} > u_n - u_{n-1} > 0$, therefore the sequence is strictly increasing.

If $\beta = \alpha + q$ the sequence follows $u_{n+1} - 2u_n + u_{n-1} = 0$ and so $u_{n+1} - u_n = u_n - u_{n-1}$ for all n (which is still increasing - it's an arithmetic progression with common difference $\beta - \alpha$).

Question (2012 STEP III Q2)

In this question, $|x| < 1$ and you may ignore issues of convergence.

(i) Simplify

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}),$$

where n is a positive integer, and deduce that

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.$$

Deduce further that

$$\ln(1-x) = -\sum_{r=0}^{\infty} \ln\left(1+x^{2^r}\right),$$

and hence that

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots.$$

(ii) Show that

$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots.$$

(i)

$$\begin{aligned}
&(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) \\
&\quad = (1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) \\
&\quad = (1-x^4)(1+x^4)\cdots(1+x^{2^n}) \\
&\quad = 1 - x^{2^{n+1}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{1-x} - \frac{x^{2^{n+1}}}{1-x} = (1+x)(1+x^2)\cdots(1+x^{2^n}) \\
\Rightarrow \quad &\frac{1}{1-x} = (1+x)(1+x^2)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x} \\
\Rightarrow \quad &-\ln(1-x) = \sum_{r=0}^{\infty} \ln(1+x^{2^r})
\end{aligned}$$

$$\Rightarrow \qquad \ln(1-x) = -\sum_{r=0}^{\infty} \ln(1+x^{2^r})$$

$$\underbrace{\Rightarrow}_{\frac{\mathrm{d}}{\mathrm{d}x}} \qquad \frac{1}{1-x} = \sum_{r=0}^{\infty} \frac{2^r x^{2^r-1}}{1+x^{2^r}}$$

$$= \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots$$

(ii) Consider

$$\begin{aligned}(1+x+x^2)&(1-x+x^2)(1-x^2+x^4)\cdots(1-x^{2^n}+x^{2^{n+1}})\\ &= (1+x^2+x^4)(1-x^2+x^4)\cdots(1-x^{2^n}+x^{2^{n+1}})\\ &= (1-x^{2^{n+1}}+x^{2^{n+2}})\end{aligned}$$

Therefore,

$$\frac{1}{1+x+x^2} = (1-x+x^2)(1-x^2+x^4)\cdots(1-x^{2^n}+x^{2^{n+1}}) + \frac{x^{2^{n+1}}}{1+x+x^2} - \frac{x^{2^{n+2}}}{1+x+x^2}$$

$$\Rightarrow \quad -\ln(1+x+x^2) = \sum_{r=0}^{\infty} \ln(1-x^{2^r}+x^{2^{r+1}})$$

$$\underbrace{\Rightarrow}_{\frac{\mathrm{d}}{\mathrm{d}x}} \qquad -\frac{1+2x}{1+x+x^2} = \sum_{r=0}^{\infty} \frac{-2^r x^{2^r-1} + 2^{r+1}x^{2^{r+1}-1}}{1-x^{2^r}+x^{2^{r+1}}}$$

$$= \frac{-1+2x}{1-x+x^2} + \frac{-2x+4x^3}{1-x^2+x^4} + \frac{-4x^3+8x^7}{1-x^4+x^8} + \cdots$$

Which is the desired result when we multiply both sides by -1

Question (2012 STEP III Q8)

The sequence F_0, F_1, F_2, ... is defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geq 0$,

$$F_{n+2} = F_{n+1} + F_n\,.$$

(i) Show that $F_0 F_3 - F_1 F_2 = F_2 F_5 - F_3 F_4\,.$

(ii) Find the values of $F_n F_{n+3} - F_{n+1} F_{n+2}$ in the two cases that arise.

(iii) Prove that, for $r = 1, 2, 3, \ldots,$

$$\arctan\left(\frac{1}{F_{2r}}\right) = \arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right)$$

and hence evaluate the following sum (which you may assume converges):

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right)\,.$$

Question (2013 STEP II Q6)

In this question, the following theorem may be used.
𝔰𝔩 *Let*u_1, u_2, ... be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leq b$ for all n, where b is some fixed number) and increasing (that is, $u_n \geq u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges). The sequence u_1, u_2, ... is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \qquad (n \geq 1)\,. \qquad (*)$$

(i) Show that, for $n \geq 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}.$$

(ii) Prove, by induction or otherwise, that $1 \leq u_n \leq 2$ for all n.

(iii) Show that the sequence u_1, u_3, u_5, ... tends to a limit, and that the sequence u_2, u_4, u_6, ... tends to a limit. Find these limits and deduce that the sequence u_1, u_2, u_3, ... tends to a limit. Would this conclusion change if the sequence were defined by $(*)$ and $u_1 = 3$?

Question (2014 STEP III Q8)

The numbers $\dot{f}(r)$ satisfy $\dot{f}(r) > \dot{f}(r+1)$ for $r = 1, 2, \ldots$. Show that, for any non-negative integer n,

$$k^n(k-1)\,\dot{f}(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} \dot{f}(r) \leq k^n(k-1)\,\dot{f}(k^n)$$

where k is an integer greater than 1.

(i) By taking $\dot{f}(r) = 1/r$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1\,.$$

Deduce that the sum $\sum\limits_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $\dot{f}(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\tfrac{1}{3}\,.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers. Show that $\sigma(n) < 80$ for all n.

Question (2016 STEP II Q8)

Evaluate the integral

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad (m > \tfrac{1}{2})\,.$$

Show by means of a sketch that

$$\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x\,, \qquad (*)$$

where m and n are positive integers with $m < n$.

(i) You are given that the infinite series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges to a value denoted by E. Use $(*)$ to obtain the following approximations for E:

$$E \approx 2\,; \qquad E \approx \frac{5}{3}\,; \qquad E \approx \frac{33}{20}\,.$$

(ii) Show that, when r is large, the error in approximating $\dfrac{1}{r^2}$ by $\displaystyle\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x$ is approximately $\dfrac{1}{4r^4}$.

Given that $E \approx 1.645$, show that $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08\,.$

Question (2016 STEP III Q4) **(i)** By considering $\dfrac{1}{1+x^r} - \dfrac{1}{1+x^{r+1}}$ for $|x| \neq 1$, simplify

$$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})}\,.$$

Show that, for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{x}{1-x^2}\,.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty} (ry)((r+1)y) = 2\mathrm{e}^{-y}(2y)$$

for $y > 0$.

Hence simplify

$$\sum_{r=-\infty}^{\infty} (ry)((r+1)y)\,,$$

for $y > 0$.

Question (2017 STEP III Q1) **(i)** Prove that, for any positive integers n and r,

$$\frac{1}{\genfrac{}{}{0pt}{}{n+r}{r+1}} = \frac{r+1}{r}\left(\frac{1}{\genfrac{}{}{0pt}{}{n+r-1}{r}} - \frac{1}{\genfrac{}{}{0pt}{}{n+r}{r}}\right).$$

Hence determine

$$\sum_{n=1}^{\infty} \frac{1}{\genfrac{}{}{0pt}{}{n+r}{r+1}}\,,$$

and deduce that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\genfrac{}{}{0pt}{}{n+2}{3}} = \frac{1}{2}\,.$

(ii) Show that, for $n \geq 3$,

$$\frac{3!}{n^3} < \frac{1}{\genfrac{}{}{0pt}{}{n+1}{3}} \quad \text{and} \quad \frac{20}{\genfrac{}{}{0pt}{}{n+1}{3}} - \frac{1}{\genfrac{}{}{0pt}{}{n+2}{5}} < \frac{5!}{n^3}\,.$$

By summing these inequalities for $n \geq 3$, show that

$$\frac{115}{96} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{116}{96}\,.$$

Note: $\genfrac{}{}{0pt}{}{n}{r}$ is another notation for $\dbinom{n}{r}$.

$$\frac{r+1}{r}\left(\frac{1}{\genfrac{}{}{0pt}{}{n+r-1}{r}} - \frac{1}{\genfrac{}{}{0pt}{}{n+r}{r}}\right) = \frac{r+1}{r}\left(\frac{r!(n-1)!}{(n+r-1)!} - \frac{r!n!}{(n+r)!}\right)$$

$$
\begin{aligned}
&= \frac{(r+1)!(n-1)!}{r(n+r-1)!}\left(1-\frac{n}{n+r}\right)\\
&= \frac{(r+1)!(n-1)!}{r(n+r-1)!}\frac{r}{n+r}\\
&= \frac{(r+1)!n!}{(n+r)!}\\
&= \frac{1}{{n+r \atop r+1}}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{{n+r \atop r+1}} &= \sum_{n=1}^{\infty}\left(\frac{r+1}{r}\left(\frac{1}{{n+r-1 \atop r}}-\frac{1}{{n+r \atop r}}\right)\right)\\
&= \frac{r+1}{r}\sum_{n=1}^{\infty}\left(\frac{1}{{n+r-1 \atop r}}-\frac{1}{{n+r \atop r}}\right)\\
&= \frac{r+1}{r}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{{n+r-1 \atop r}}-\frac{1}{{n+r \atop r}}\right)\\
&= \frac{r+1}{r}\lim_{N\to\infty}\left(\frac{1}{{1+r-1 \atop r}}-\frac{1}{{N+r \atop r}}\right)\\
&= \frac{r+1}{r}\frac{1}{{1+r-1 \atop r}} \qquad \left(\text{since } \frac{1}{{N+r \atop r}} \to 0\right)\\
&= \frac{r+1}{r}
\end{aligned}
$$

When $r=2$, we have:

$$
\begin{aligned}
\frac{3}{2} &= \sum_{n=1}^{\infty}\frac{1}{{n+2 \atop 3}}\\
&= \frac{1}{{1+2 \atop 3}}+\sum_{n=2}^{\infty}\frac{1}{{n+2 \atop 3}}\\
&= 1+\sum_{n=2}^{\infty}\frac{1}{{n+2 \atop 3}}
\end{aligned}
$$

$$
\Rightarrow \qquad \sum_{n=2}^{\infty}\frac{1}{{n+2 \atop 3}} = \frac{1}{2}
$$

$$
\begin{aligned}
\frac{1}{{n+1 \atop 3}} &= \frac{3!}{(n+1)n(n-1)}\\
&= \frac{3!}{n^3-n}\\
&> \frac{3!}{n^3}
\end{aligned}
$$

$$
\frac{20}{{n+1 \atop 3}}-\frac{1}{{n+2 \atop 5}} = \frac{5!}{(n+1)n(n-1)}-\frac{5!}{(n+2)(n+1)n(n-1)(n-2)}
$$

$$\begin{aligned}
&= \frac{5!}{n^3}\frac{n^2}{n^2-1}\left(1-\frac{1}{n^2-4}\right)\\
&= \frac{5!}{n^3}\frac{n^2}{n^2-1}\left(\frac{n^2-5}{n^2-4}\right)\\
&= \frac{5!}{n^3}\frac{n^2(n^2-5)}{(n^2-1)(n^2-4)}\\
&< \frac{5!}{n^3}
\end{aligned}$$

Since $k(k-5) < (k-1)(k-4) \Leftrightarrow 0 < 4$, this only makes sense if $n \geq 3$

$$\begin{aligned}
& & \frac{3!}{n^3} &< \frac{1}{\genfrac{}{}{0pt}{}{n+1}{3}} & \text{(if } n \geq 3)\\
\Rightarrow & & \sum_{n=3}^{\infty}\frac{3!}{n^3} &< \sum_{n=3}^{\infty}\frac{1}{\genfrac{}{}{0pt}{}{n+1}{3}}\\
\Rightarrow & & \frac{6}{1^3}+\frac{6}{2^3}+\sum_{n=3}^{\infty}\frac{3!}{n^3} &< \frac{6}{1^3}+\frac{6}{2^3}+\sum_{n=3}^{\infty}\frac{1}{\genfrac{}{}{0pt}{}{n+1}{3}}\\
\Rightarrow & & \sum_{n=1}^{\infty}\frac{3!}{n^3} &< 6+\frac{3}{4}+\sum_{n=2}^{\infty}\frac{1}{\genfrac{}{}{0pt}{}{n+2}{2+1}}\\
\Rightarrow & & \sum_{n=1}^{\infty}\frac{3!}{n^3} &< 6+\frac{3}{4}+\frac{1}{2}=\frac{29}{4}\\
\Rightarrow & & \sum_{n=1}^{\infty}\frac{1}{n^3} &< \frac{29}{24}=\frac{116}{96}
\end{aligned}$$

$$\begin{aligned}
& & \frac{20}{\genfrac{}{}{0pt}{}{n+1}{3}}-\frac{1}{\genfrac{}{}{0pt}{}{n+2}{5}} &< \frac{5!}{n^3}\\
\Rightarrow & & \sum_{n=3}^{\infty}\left(\frac{20}{\genfrac{}{}{0pt}{}{n+1}{3}}-\frac{1}{\genfrac{}{}{0pt}{}{n+2}{5}}\right) &< \sum_{n=3}^{\infty}\frac{5!}{n^3}\\
\Rightarrow & & \frac{120}{1^3}+\frac{120}{2^3}+\sum_{n=3}^{\infty}\frac{20}{\genfrac{}{}{0pt}{}{n+1}{3}}-\sum_{n=3}^{\infty}\frac{1}{\genfrac{}{}{0pt}{}{n+2}{5}} &< \frac{120}{1^3}+\frac{120}{2^3}+\sum_{n=3}^{\infty}\frac{5!}{n^3}\\
\Rightarrow & & \frac{120}{1^3}+\frac{120}{2^3}+\sum_{n=2}^{\infty}\frac{20}{\genfrac{}{}{0pt}{}{n+2}{2+1}}-\sum_{n=1}^{\infty}\frac{1}{\genfrac{}{}{0pt}{}{n+4}{4+1}} &< \frac{120}{1^3}+\frac{120}{2^3}+\sum_{n=3}^{\infty}\frac{5!}{n^3}\\
\Rightarrow & & \frac{120}{1^3}+\frac{120}{2^3}+\frac{20}{2}-\frac{4+1}{4} &< \sum_{n=1}^{\infty}\frac{5!}{n^3}\\
\Rightarrow & & \frac{115}{96} &< \sum_{n=1}^{\infty}\frac{1}{n^3}
\end{aligned}$$

Question (2017 STEP III Q8)

Prove that, for any numbers a_1, a_2, ..., and b_1, b_2, ..., and for $n \geq 1$,

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) = a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m).$$

(i) By setting $b_m = \sin mx$, show that

$$\sum_{m=1}^{n} \cos(m + \tfrac{1}{2})x = \tfrac{1}{2}\big(\sin(n+1)x - \sin x\big)\tfrac{1}{2}x.$$

Note: $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$

(ii) Show that

$$\sum_{m=1}^{n} m \sin mx = \big(p\sin(n+1)x + q\sin nx\big)^2 \tfrac{1}{2}x,$$

where p and q are to be determined in terms of n.

Note: $2\sin A \sin B = \cos(A-B) - \cos(A+B)$;
$2\cos A \sin B = \sin(A+B) - \sin(A-B)$.

Question (1987 STEP I Q4)

Show that the sum of the infinite series

$$\log_2 e - \log_4 e + \log_{16} e - \ldots + (-1)^n \log_{2^{2^n}} e + \ldots$$

is

$$\frac{1}{\ln(2\sqrt{2})}.$$

[$\log_a b = c$ is equivalent to $a^c = b$.]

Let $S = \log_2 e - \log_4 e + \log_{16} e - \ldots + (-1)^n \log_{2^{2^n}} e + \ldots$ then

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} (-1)^n \log_{2^{2^n}} e \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\log e}{\log 2^{2^n}} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\log e}{2^n \log 2} \\
&= \frac{\log e}{\log 2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \\
&= \frac{1}{\log_e 2} \frac{1}{1 + \frac{1}{2}}
\end{aligned}$$

$$= \frac{1}{\ln(2^{3/2})}$$
$$= \frac{1}{\ln(2\sqrt{2})}$$

Question (1987 STEP II Q5)

If $y = \mathrm{f}(x)$, then the inverse of f (when it exists) can be obtained from *Lagrange's identity*. This identity, which you may use without proof, is

$$\mathrm{f}^{-1}(y) = y + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} \left[y - \mathrm{f}\left(y\right)\right]^n ,$$

provided the series converges.

(i) Verify Lagrange's identity when $\mathrm{f}(x) = \alpha x$, $(0 < \alpha < 2)$.

(ii) Show that one root of the equation

$$\tfrac{1}{2} = x - \tfrac{1}{4}x^3$$

is

$$x = \sum_{n=0}^{\infty} \frac{(3n)!}{n!\,(2n+1)!2^{4n+1}}$$

(iii) Find a solution for x, as a series in λ, of the equation

$$x = \mathrm{e}^{\lambda x}.$$

[You may assume that the series in part **(ii)** converges, and that the series in part **(iii)** converges for suitable λ.]

(i) If $f(x) = \alpha x$ then $f^{-1}(x) = \frac{1}{\alpha}x$.

$$\begin{aligned}
\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y - \alpha y]^n &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[(1-\alpha)^n y^n] \\
&= (1-\alpha)^n n! y \\
\Rightarrow \quad y + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y - \alpha y]^n &= y + \sum_{n=1}^{\infty} (1-\alpha)^n y \\
&= y + y\left(\frac{1}{1-(1-\alpha)} - 1\right) \\
&= \frac{1}{\alpha} y
\end{aligned}$$

Where we can sum the geometric progression if $|1-\alpha| < 1 \Leftrightarrow 0 < \alpha < 2$

(ii) Suppose that $f(x) = x - \frac{1}{4}x^3$. We would like to find $f^{-1}(\frac{1}{2})$.

$$\begin{aligned}
\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y-(y+\frac{1}{4}y^3)]^n &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[\frac{1}{4^n}y^{3n}] \\
&= \frac{1}{4^n}\frac{(3n)!}{(2n+1)!}y^{2n+1} \\
\Rightarrow \qquad f^{-1}(\frac{1}{2}) &= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{4^n}\frac{(3n)!}{n!(2n+1)!}\frac{1}{2^{2n+1}} \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(3n)!}{n!(2n+1)!}\frac{1}{2^{4n+1}}
\end{aligned}$$

Since when $n=0$ $\frac{0!}{0!1!}\frac{1}{2^{0+1}}=\frac{1}{2}$ we can include the wayward $\frac{1}{2}$ in our infinite sum and so we have the required result.

(iii) Consider $f(x)=x-e^{\lambda x}$ we are interested in $f^{-1}(0)$.

$$\begin{aligned}
\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y-(y-e^{\lambda y})]^n &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[e^{n\lambda y}] \\
&= n^{n-1}\lambda^{n-1}e^{n\lambda y} \\
\Rightarrow \qquad f^{-1}(0) &= \sum_{n=1}^{\infty}\frac{1}{n!}n^{n-1}\lambda^{n-1}
\end{aligned}$$

We don't care about convergence, but it's worth noting this has a radius of convergence of $\frac{1}{e}$ (ie this series is valid if $|\lambda|<\frac{1}{e}$).

Question (1987 STEP III Q7)

Prove that

$$\tan^{-1}t = t-\frac{t^3}{3}+\frac{t^5}{5}-\cdots+\frac{(-1)^nt^{2n+1}}{2n+1}+(-1)^{n+1}\int_0^t\frac{x^{2n+2}}{1+x^2}\,\mathrm{d}x.$$

Hence show that, if $0\leqslant t\leqslant 1$, then

$$\frac{t^{2n+3}}{2(2n+3)}\leqslant\left|\tan^{-1}t-\sum_{r=0}^{n}\frac{(-1)^rt^{2r+1}}{2r+1}\right|\leqslant\frac{t^{2n+3}}{2n+3}.$$

Show that, as $n\to\infty$,

$$4\sum_{r=0}^{n}\frac{(-1)^r}{(2r+1)}\to\pi,$$

but that the error in approximating π by $4\sum_{r=0}^{n}\frac{(-1)^r}{(2r+1)}$ is at least 10^{-2} if n is less than or equal to 98.

We start by noticing that $\tan^{-1} t = \displaystyle\int_0^t \frac{1}{1+x^2}\mathrm{d}x$.

Consider the geometric series $1 - x^2 + (-x^2)^2 + \cdots + (-x^2)^n = \frac{1-(-x^2)^{n+1}}{1+x^2}$. Therefore, $(1+x^2)(1 - x^2 + (-x^2)^2 + \cdots + (-x^2)^n) = 1 - (-x^2)^{n+1}$ or
$1 = (1+x^2)(1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}) + (-1)^{n+1} x^{2n+2}$

$$\begin{aligned}
\tan^{-1} t &= \int_0^t \frac{1}{1+x^2}\mathrm{d}x \\
&= \int_0^t \frac{(1+x^2)(1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}) + (-1)^{n+1} x^{2n+2}}{x^2+1}\mathrm{d}x \\
&= \int_0^t (1 - x^2 + x^4 - \cdots + (-1)^n x^{2n})\mathrm{d}x + \int_0^t \frac{(-1)^{n+1} x^{2n+2}}{x^2+1}\mathrm{d}x \\
&= t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + (-1)^n \frac{t^{2n+1}}{2n+1} + \int_0^t \frac{(-1)^{n+1} x^{2n+2}}{x^2+1}\mathrm{d}x \\
&= \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1} + \int_0^t \frac{(-1)^{n+1} x^{2n+2}}{x^2+1}\mathrm{d}x
\end{aligned}$$

Therefore we can say (for $0 \le t \le 1$)

$$\begin{aligned}
\left|\tan^{-1} t - \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1}\right| &= \left|\int_0^t \frac{(-1)^{n+1} x^{2n+2}}{x^2+1}\mathrm{d}x\right| \\
&\le \left|\int_0^t x^{2n+2}\mathrm{d}x\right| \\
&= \frac{t^{2n+3}}{2n+3}
\end{aligned}$$

$$\begin{aligned}
\left|\tan^{-1} t - \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1}\right| &= \left|\int_0^t \frac{(-1)^{n+1} x^{2n+2}}{x^2+1}\mathrm{d}x\right| \\
&\ge \left|\int_0^t \frac{(-1)^{n+1} x^{2n+2}}{1+1}\mathrm{d}x\right| \\
&= \frac{t^{2n+3}}{2(2n+3)}
\end{aligned}$$

Since $\tan^{-1} 1 = \frac{\pi}{4}$ we must have that:

$$\lim_{n\to\infty} \left|\frac{\pi}{4} - \sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)}\right| \to 0 \Rightarrow \lim_{n\to\infty} 4\sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)} \to \pi$$

However,

$$\left|4\sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)} - \pi\right| \ge 4\frac{1}{2(2n+3)}$$

$$= \frac{2}{2n+3}$$

$$\begin{aligned} & \frac{2}{2n+3} \geq 10^{-2} \\ \Leftrightarrow \quad & 200 \geq 2n+3 \\ \Leftrightarrow \quad & 197 \geq 2n \\ \Leftrightarrow \quad & 98.5 \geq n \end{aligned}$$

Therefore we need more than 98 terms to get two decimal places of accuracy. Not great!

Question (1991 STEP III Q10)
The equation

$$x^n - qx^{n-1} + r = 0,$$

where $n \geqslant 5$ and q and r are real constants, has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. The sum of the products of m distinct roots is denoted by Σ_m (so that, for example, $\Sigma_3 = \sum \alpha_i \alpha_j \alpha_k$ where the sum runs over the values of i, j and k with $n \geqslant i > j > k \geqslant 1$). The sum of mth powers of the roots is denoted by S_m (so that, for example, $S_3 = \sum_{i=1}^{n} \alpha_i^3$).
Prove that $S_p = q^p$ for $1 \leqslant p \leqslant n-1$.
You may assume that for any nth degree equation and $1 \leqslant p \leqslant n$

$$S_p - S_{p-1}\Sigma_1 + S_{p-2}\Sigma_2 - \cdots + (-1)^{p-1} S_1 \Sigma_{p-1} + (-1)^p p \Sigma_p = 0.]$$

Find expressions for S_n, S_{n+1} and S_{n+2} in terms of q, r and n. Suggest an expression for S_{n+m}, where $m < n$, and prove its validity by induction.

Claim: $S_p = q^p$ for $1 \leq p \leq n-1$
Proof: When $p = 1$, $S_p = \Sigma_1 = q$ as expected.
Note that $\Sigma_i = 0$ for $i = 2, \cdots, n-1$.
Using $S_p = S_{p-1}\Sigma_1 - S_{p-2}\Sigma_2 + \cdots + (-1)^{p-1+1} S_1 \Sigma_{p-1} + (-1)^{p+1} p \Sigma_p$, we can see that $S_p = qS_{p-q}$ when $1 \leq p \leq n-1$, ie $S_p = q^p$.
Note that

$$\begin{aligned} S_n &= \sum \alpha_i^n \\ &= q \sum \alpha_i^{n-1} - \sum r \\ &= qS_{n-1} - nr \\ &= q^n - nr \end{aligned}$$

$$\begin{aligned} S_{n+1} &= \sum \alpha_i^{n+1} \\ &= q \sum \alpha_i^n - r \sum \alpha_i \\ &= q^{n+1} - rq \end{aligned}$$

$$S_{n+2} = \sum \alpha_i^{n+2}$$

$$\begin{aligned} &= q\sum \alpha_i^{n+1} - r\sum \alpha_i^2 \\ &= q^{n+2} - rq^2 \end{aligned}$$

Claim: $S_{n+m} = q^{n+m} - rq^m$
Proof: The obvious

Question (1992 STEP II Q7)
The cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a, b and c. Express p, q and r in terms of a, b and c.

(i) If $p = 0$ and two of the roots are equal to each other, show that

$$4q^3 + 27r^2 = 0.$$

(ii) Show that, if two of the roots of the original equation are equal to each other, then

$$4\left(q - \frac{p^2}{3}\right)^3 + 27\left(\frac{2p^3}{27} - \frac{pq}{3} + r\right)^2 = 0.$$

$p = a + b + c, q = ab + bc + ca, r = abc$

(i) Suppose two roots are equal to each other, this means that one of the roots is also a root of the derivative. ie

$$\begin{aligned} 0 &= x^3 + qx - r \\ 0 &= 3x^2 + q \end{aligned}$$

have a common root, but this root must satisfy $x^2 = -\frac{q}{3}$. Then

$$\begin{aligned} 0 &= x^3 + qx - r \\ &= x^3 - 3x^3 - r \\ &= -2x^3 - r \\ \Rightarrow \quad r^2 &= 4x^6 \\ &= 4\left(-\frac{q}{3}\right)^3 \\ \Rightarrow \quad 0 &= 27r^2 + 4q^3 \end{aligned}$$

(ii) Consider $x = z + \frac{p}{3}$, then the equation is:

$$\begin{aligned} x^3 - px^2 + qx - r &= (z + \frac{p}{3})^3 - p(z + \frac{p}{3})^2 + q(z + \frac{p}{3}) - r \\ &= z^3 + pz^2 + \frac{p^2}{3}z + \frac{p^3}{27} - \end{aligned}$$

$$\begin{aligned}
&- pz^2 - \frac{2p^2}{3} z - \frac{p^3}{9} + \\
&qz + \frac{pq}{3} - r \\
&= z^3 + \left(\frac{p^2}{3} - \frac{2p^2}{3} + q \right) z + \left(\frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} - r \right) \\
&= z^3 + \left(-\frac{p^2}{3} + q \right) z + \left(-\frac{2p^3}{27} + \frac{pq}{3} - r \right)
\end{aligned}$$

Since this equation must also have repeated roots we must have:

$$4 \left(-\frac{p^2}{3} + q \right)^3 + 27 \left(-\frac{2p^3}{27} + \frac{pq}{3} - r \right)^2 = 0$$

which is exactly our desired result

Question (1996 STEP III Q7) **(i)** If $x+y+z = \alpha$, $xy+yz+zx = \beta$ and $xyz = \gamma$, find numbers A, B and C such that

$$x^3 + y^3 + z^3 = A\alpha^3 + B\alpha\beta + C\gamma.$$

Solve the equations

$$\begin{aligned}
x + y + z &= 1 \\
x^2 + y^2 + z^2 &= 3 \\
x^3 + y^3 + z^3 &= 4.
\end{aligned}$$

(ii) The area of a triangle whose sides are a, b and c is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter $\frac{1}{2}(a+b+c)$. If a, b and c are the roots of the equation

$$x^3 - 16x^2 + 81x - 128 = 0,$$

find the area of the triangle.

(i)

$$\begin{aligned}
(x+y+z)^3 &= x^3 + y^3 + z^3 + \\
&\quad 3xy^2 + 3xz^2 + 3yx^2 + \cdots + 3zy^2 \\
&\quad + 6xyz \\
(x+y+z)(xy+yz+zx) &= x^2y + x^2z + \cdots + z^2x + 3xyz \\
x^3 + y^3 + z^3 &= (x+y+z)^3 - 3(xy^2 + \cdots + zy^2) - 6xyz \\
&= \alpha^3 - 3(\alpha\beta - 3\gamma) - 6\gamma
\end{aligned}$$

$$= \alpha^3 - 3\alpha\beta + 3\gamma$$

Since $4 = 1^3 - 3 \cdot 1 \cdot (-1) + 3\gamma \Rightarrow \gamma = 0$, therefore one of $x, y, z = 0$. WLOG $x = 0$, so

$y + z = 1, y^2 + z^2 = 3 \Rightarrow y^2 + (1-y)^2 = 3 \Rightarrow y^2 - y - 1 = 0 \Rightarrow y = \frac{1 \pm \sqrt{5}}{2}$, so we have

$(x, y, z) = (0, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ and permutations.

(ii)

$$A^2 = s(s-a)(s-b)(s-c)$$

Notice the second part is the same as plugging $s = 16/2 = 8$ into our polynomial

Therefore

$$\begin{aligned} A^2 &= 8 \cdot (8^3 - 16 \cdot 8^2 + 81 \cdot 8 - 128) \\ &= 8 \cdot 8(8^2 - 16 \cdot 8 + 81 - 16) \\ &= 64(-64 + 81 - 16) \\ &= 64 \end{aligned}$$

Therefore $A = 8$

Question (1997 STEP III Q4)

In this question, you may assume that if $k_1, \dots, k_n$ are distinct positive real numbers, then

$$\frac{1}{n}\sum_{r=1}^{n} k_r > \left(\prod_{r=1}^{n} k_r\right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean. Suppose that a, b, c and d are positive real numbers such that the polynomial

$$\mathrm{f}(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots.

(i) Show that pqr, qrs, rsp and spq are distinct, where p, q, r and s are the roots of the polynomial f.

(ii) By considering the relationship between the coefficients of f and its roots, show that $c > d$.

(iii) Explain why the polynomial $\mathrm{f}'(x)$ must have three distinct roots.

(iv) By differentiating f, show that $b > c$.

(v) Show that $a > b$.

(i) Suppose $pqr = qrs$, since the roots are positive, we can divide by qr to obtain $p = s$ (a contradiction. Therefore all those terms are distinct.

(ii) $4c^3 = pqr + qrs + rsp + spq$, $d^4 = pqrs$.

Applying AM-GM, we obtain:

$$c^3 = \frac{pqr + qrs + rsp + spq}{4} > \sqrt[4]{p^3q^3r^3s^3} = d^3$$

$$\Rightarrow \qquad c > d$$

(iii) There must be a turning point between each root (since there are no repeated roots).

(iv) $f'(x) = 4x^3 - 12ax^2 + 12b^2 - 4c^3 = 4(x^3 - 3ax^2 + 3b^2 - c^3)$. Letting the roots of this polynomial be α, β, γ and again applying AM-GM, we must have:

$$b^2 = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{3} > \sqrt[3]{\alpha^2\beta^2\gamma^2} = c^2$$

$$\Rightarrow \qquad b > c$$

(v) Again, since there are turning points between the roots of $f'(x)$ we must have distinct roots for $f''(x)$, ie:

$f''(x) = 3x^2 - 6ax + 6b^2 = 3(x^2 - 2ax + b^2)$ has distinct real roots. But for this to occur we must have that $(2a)^2 - 4b^2 = 4(a^2 - b^2) > 0$, ie $a > b$

Question (2007 STEP III Q1)

In this question, do not consider the special cases in which the denominators of any of your expressions are zero. Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of t_i, where $t_1 = \tan\theta_1$, etc. Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of a, b, c, d and e for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$. The four real numbers θ_1, θ_2, θ_3 and θ_4 lie in the range $0 \leq \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0\,,$$

where p and α are independent of θ. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.

$$\begin{aligned}
\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) &= \frac{\tan(\theta_1 + \theta_2) + \tan(\theta_3 + \theta_4)}{1 - \tan(\theta_1 + \theta_2)\tan(\theta_3 + \theta_4)} \\
&= \frac{\frac{t_1 + t_2}{1 - t_1t_2} + \frac{t_3 + t_4}{1 - t_3t_4}}{1 - \frac{t_1 + t_2}{1 - t_1t_2}\frac{t_3 + t_4}{1 - t_3t_4}} \\
&= \frac{(t_1 + t_2)(1 - t_3t_4) + (t_3 + t_4)(1 - t_1t_2)}{(1 - t_1t_2)(1 - t_3t_4) - (t_1 + t_2)(t_3 + t_4)} \\
&= \frac{t_1 + t_2 + t_3 + t_4 - (t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)}{1 - t_1t_2 - t_1t_3 - t_1t_4 - t_2t_3 - t_2t_4 - t_3t_4}
\end{aligned}$$

If t_1, t_2, t_3, t_4 are the roots of $at^4 + bt^3 + ct^2 + dt + e = 0$, then $t_1 + t_2 + t_3 + t_4 = -\frac{b}{a}$, $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{c}{a}$, $t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 = -\frac{d}{a}$, therefore the expression is:

$$\begin{aligned}
\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) &= \frac{-\frac{b}{a} + \frac{d}{a}}{1 - \frac{c}{a}} \\
&= \frac{d - b}{a - c}
\end{aligned}$$

$$\begin{aligned}
0 &= p\cos 2\theta + \cos(\theta - \alpha) + p \\
&= p(2\cos^2\theta - 1) + \cos\theta\cos\alpha - \sin\theta\sin\alpha + p \\
&= 2p\cos^2\theta + \cos\theta\cos\alpha - \sin\theta\sin\alpha \\
\Rightarrow \qquad 0 &= 2p\cos\theta + \cos\alpha - \tan\theta\sin\alpha \\
\Rightarrow \qquad -2p\cos\theta &= \cos\alpha - \tan\theta\sin\alpha
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & 4p^2\cos^2\theta = \cos^2\alpha - 2\sin\alpha\cos\alpha\tan\theta + \sin^2\alpha\tan^2\theta \\
& 4p^2\frac{1}{1+\tan^2\theta} = \cos^2\alpha - \sin 2\alpha\tan\theta + \sin^2\alpha\tan^2\theta \\
\Rightarrow \quad & 4p^2 = \cos^2\alpha - \sin 2\alpha t + t^2 - \sin 2\alpha t^3 + \sin^2\alpha t^4 \\
\Rightarrow \quad & \tan(\theta_1+\theta_2+\theta_3+\theta_4) = \frac{0}{\sin^2\alpha - 1} \\
& = 0 \\
\Rightarrow \quad & \theta_1+\theta_2+\theta_3+\theta_4 = n\pi
\end{aligned}$$

Question (2008 STEP III Q1)

Find all values of a, b, x and y that satisfy the simultaneous equations

$$\begin{aligned}
a + b &= 1 \\
ax + by &= \tfrac{1}{3} \\
ax^2 + by^2 &= \tfrac{1}{5} \\
ax^3 + by^3 &= \tfrac{1}{7}\,.
\end{aligned}$$

[**H**int: you may wish to start by multiplying the second equation by $x + y$.]

This is a second order recurrence relation, so we need to find m and n such that;

$$\begin{aligned}
& \frac{1}{5} = m\frac{1}{3} + n \\
& \frac{1}{7} = m\frac{1}{5} + n\frac{1}{3} \\
\Rightarrow \quad & m, n = \frac{6}{7}, -\frac{3}{35}
\end{aligned}$$

So we now need to solve the characteristic equation:
$\lambda^2 - \frac{6}{7}\lambda + \frac{3}{35} = 0$
So $x, y = \frac{15 \pm 2\sqrt{30}}{35}$.
We need,

$$\begin{aligned}
& 1 = a + b \\
& \frac{1}{3} = a\frac{15+2\sqrt{30}}{35} + b\frac{15-2\sqrt{30}}{35} \\
& \frac{1}{3} = \frac{15}{35} + \frac{2\sqrt{30}}{35}(a-b) \\
\Rightarrow \quad & -\frac{\sqrt{30}}{18} = a - b \\
\Rightarrow \quad & a = \frac{18-\sqrt{30}}{36} \\
& b = \frac{18+\sqrt{30}}{38}
\end{aligned}$$

So our two answers are:

$$(a,b,x,y) = \left(\frac{18 \pm \sqrt{30}}{36}, \frac{18 \mp \sqrt{30}}{36}, \frac{15 \pm 2\sqrt{30}}{35}, \frac{15 \mp 2\sqrt{30}}{35}, \right)$$

Question (2009 STEP III Q5)

The numbers x, y and z satisfy

$$\begin{aligned} x+y+z &= 1 \\ x^2+y^2+z^2 &= 2 \\ x^3+y^3+z^3 &= 3\,. \end{aligned}$$

Show that

$$yz+zx+xy = -\frac{1}{2}\,.$$

Show also that $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y = -1$, and hence that

$$xyz = \frac{1}{6}\,.$$

Let $S_n = x^n+y^n+z^n$. Use the above results to find numbers a, b and c such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2}\,,$$

holds for all n.

$$\begin{aligned} & (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \\ \Rightarrow \quad & 1^2 = 2+2(xy+yz+zx) \\ \Rightarrow \quad & xy+yz+zx = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} 1\cdot 2 &= (x+y+z)(x^2+y^2+z^2) \\ &= x^3+y^3+z^3+x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \\ &= 3+x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \\ \Rightarrow \quad -1 &= x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \end{aligned}$$

$$\begin{aligned} (x+y+z)^3 &= x^3+y^3+z^3+ \\ & \quad 3xy^2+3xz^2+\cdots+3zx^2+3zy^2+ \\ & \quad 6xyz \\ \Rightarrow \quad 1 &= 3+3(-1)+6xyz \\ \Rightarrow \quad xyz &= \frac{1}{6} \end{aligned}$$

Since we have $f(t) = (t-x)(t-y)(t-z) = t^3-t^2-\frac{1}{2}t-\frac{1}{6}$ is zero for x,y,z we can notice that:

$t^{n+1} = t^n + \frac{1}{2}t^{n-1} + \frac{1}{6}t^{n-2}$ is also true for x,y,z (by multiplying by t^{n-2}.

Therefore:

$S_{n+1} = S_n + \frac{1}{2}S_{n-1} + \frac{1}{6}S_{n-2}$

Question (2014 STEP III Q1)

Let a, b and c be real numbers such that $a + b + c = 0$ and let

$$(1 + ax)(1 + bx)(1 + cx) = 1 + qx^2 + rx^3$$

for all real x. Show that $q = bc + ca + ab$ and $r = abc$.

(i) Show that the coefficient of x^n in the series expansion (in ascending powers of x) of $\ln(1 + qx^2 + rx^3)$ is $(-1)^{n+1}S_n$ where

$$S_n = \frac{a^n + b^n + c^n}{n}, \qquad (n \geq 1).$$

(ii) Find, in terms of q and r, the coefficients of x^2, x^3 and x^5 in the series expansion (in ascending powers of x) of $\ln(1 + qx^2 + rx^3)$ and hence show that $S_2S_3 = S_5$.

(iii) Show that $S_2S_5 = S_7$.

(iv) Give a proof of, or find a counterexample to, the claim that $S_2S_7 = S_9$.

$$\begin{aligned}(1 + ax)(1 + bx)(1 + cx) &= (1 + (a + b)x + abx^2)(1 + cx)\\ &= 1 + (a + b + c)x + (ab + bc + ca)x^2 + abcx^3\end{aligned}$$

Therefore by comparing coefficients, $q = bc + ca + ab$ and $r = abc$ as required.

(i)

$$\begin{aligned}\ln(1 + qx^2 + rx^3) &= \ln(1 + ax) + \ln(1 + bx) + \ln(1 + cx)\\ &= -\sum_{n=1}^{\infty} \frac{(-ax)^n}{n} - \sum_{n=1}^{\infty} \frac{(-bx)^n}{n} - \sum_{n=1}^{\infty} \frac{(-cx)^n}{n}\\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a^n + b^n + c^n)}{n} x^n\\ &= \sum_{n=1}^{\infty} (-1)^{n+1} S_n x^n\end{aligned}$$

(ii)

$$\begin{aligned}\ln(1 + qx^2 + rx^3) &= (qx^2 + rx^3) - \frac{(qx^2 + rx^3)^2}{2} + O(x^6)\\ &= qx^2 + rx^3 - \frac{1}{2}q^2x^4 - qrx^5 + O(x^6)\end{aligned}$$

Comparing coefficients we see that $S_2 = -q$ and $S_3 = r$, we also must have $S_5 = -qr = S_2S_3$ as required.

(iii)

$$\begin{aligned}\ln(1+qx^2+rx^3) &= (qx^2+rx^3) - \frac{(qx^2+rx^3)^2}{2} + \frac{(qx^2+rx^3)^3}{3} + O(x^8) \\ &= qx^2 + rx^3 - \frac{1}{2}q^2x^4 - qrx^5 + \frac{1}{2}rx^6 + \frac{1}{3}q^3x^6 + q^2rx^7 + O(x^8) \\ &= qx^2 + rx^3 - \frac{1}{2}q^2x^4 - qrx^5 + \left(\frac{1}{2}r + \frac{1}{3}q^3\right)x^6 + q^2rx^7\end{aligned}$$

Comparing coefficients we see that $S_2 = -q$ and $S_5 = -qr$, we also must have $S_7 = q^2r = S_2S_5$ as required.

(iv) Let $a = b = 1, c = -2$, then $S_2 = \frac{1^2+1^2+(-2)^2}{2} = 3, S_7 = \frac{1^2+1^2+(-2)^7}{7} = -18, S_9 = \frac{1^1+1^2+(-2)^9}{9} = \frac{2-512}{9} \neq 3\cdot(-18)$

Question (2015 STEP III Q6) **(i)** Let w and z be complex numbers, and let $u = w+z$ and $v = w^2+z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leq 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$\begin{aligned} w+z-u &= 0 \\ w^2+z^2-u^2 &= -\tfrac{2}{3} \\ w^3+z^3-\lambda u &= -\lambda \end{aligned}$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u, all real.

Are w and z necessarily real? Give a proof or counterexample.

(i) Notice that $u^2 = v+2wz$, so w, z are roots of the quadratic $t^2 - ut + \frac{u^2-v}{2}$. Therefore they are both real if $u^2 \geq 2(u^2-v) \Rightarrow 2v \geq u^2$.

(ii)

$$\begin{aligned} w+z &= u \\ w^2+z^2 &= u^2 - \tfrac{2}{3} \\ w^3+z^3 &= \lambda(u-1) \end{aligned}$$

$$wz = \frac{u^2-(u^2-\frac{2}{3})}{2} = \tfrac{1}{3}$$

$$\begin{aligned} (w+z)(w^2+z^2) &= w^3+z^3+wz(w+z) \\ u(u^2-\tfrac{2}{3}) &= \lambda(u-1) + \frac{1}{3}u \\ \Rightarrow \quad u^3 - u &= \lambda(u-1) \end{aligned}$$

$$\Rightarrow \qquad 0 = (u-1)(u(u+1)-\lambda)$$
$$\Rightarrow \qquad 0 = (u-1)(u^2+u-\lambda)$$

Therefore there will be at most 3 values for u, unless 1 is a root of $u^2+u-\lambda$, ie $\lambda = 2$.

Suppose $u = 1$, then we have:

$w+z = 1, wz = 1/3 \Rightarrow w, z = \frac{-1\pm\sqrt{-1/3}}{2}$ which are clearly complex.

Question (2017 STEP III Q3)

Let α, β, γ and δ be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0\,.$$

You are given that, for any such equation, $\alpha\beta+\gamma\delta$, $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$ satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr-4s)y + (4qs - p^2s - r^2) = 0\,.$$

Determine A. Now consider the quartic equation given by $p = 0$, $q = 3$, $r = -6$ and $s = 10$.

(i) Find the value of $\alpha\beta+\gamma\delta$, given that it is the largest root of the corresponding cubic equation.

(ii) Hence, using the values of q and s, find the value of $(\alpha+\beta)(\gamma+\delta)$ and the value of $\alpha\beta$ given that $\alpha\beta > \gamma\delta$.

(iii) Using these results, and the values of p and r, solve the quartic equation.

$$\begin{aligned} A &= -(\alpha\beta+\gamma\delta+\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma) \\ &= -q \end{aligned}$$

(i) The corresponding cubic equation is:

$$\begin{aligned} 0 &= y^3 - 3y^2 - 40y + (120-36) \\ &= y^3 - 3y^2 - 40y + 84 \\ &= (y-7)(y-2)(y+6) \end{aligned}$$

Therefore $\alpha\beta+\gamma\delta = 7$

(ii)

$$(\alpha+\beta)(\gamma+\delta) = \alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta$$

$$= 3 - (\alpha\beta + \gamma\delta)$$
$$= 3 - 7 = -4$$

Let $\alpha\beta$ and $\gamma\delta$ be the roots of a quadratic; then the quadratic will be $t^2 - 7t + 10 = 0 \Rightarrow t = 2, 5$ so $\alpha\beta = 5$

(iii) $\alpha\beta = 5, \gamma\delta = 2$

Consider the quadratic with roots $\alpha + \beta$ and $\gamma + \delta$, then

$t^2 - 4 = 0 \Rightarrow t = \pm 2$.

Suppose $\alpha + \beta = 2, \gamma + \delta = -2$ then

$\alpha, \beta = 1 \pm 2i, \gamma, \delta = -1 \pm i$

$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 5\gamma + 2\beta + 2\alpha + 5\delta = -6 \neq 6$

Suppose $\alpha + \beta = -2, \gamma + \delta = 2$ then

$\alpha, \beta = -1 \pm 2i, \gamma, \delta = 1 \pm i$

$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 5\gamma + 2\beta + 2\alpha + 5\delta = 6$, therefore these are there roots. (In some order):

$1 \pm i, -1 \pm 2i$

Question (2018 STEP III Q1) **(i)** The function f is given by

$$f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

Find the stationary point of the curve $y = f(\beta)$ and sketch the curve. Sketch also the curve $y = g(\beta)$, where

$$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \qquad (\beta \neq 0)\,.$$

(ii) Let u and v be the roots of the equation

$$x^2 + \alpha x + \beta = 0\,,$$

where $\beta \neq 0$. Obtain expressions in terms of α and β for $u + v + \dfrac{1}{uv}$ and $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

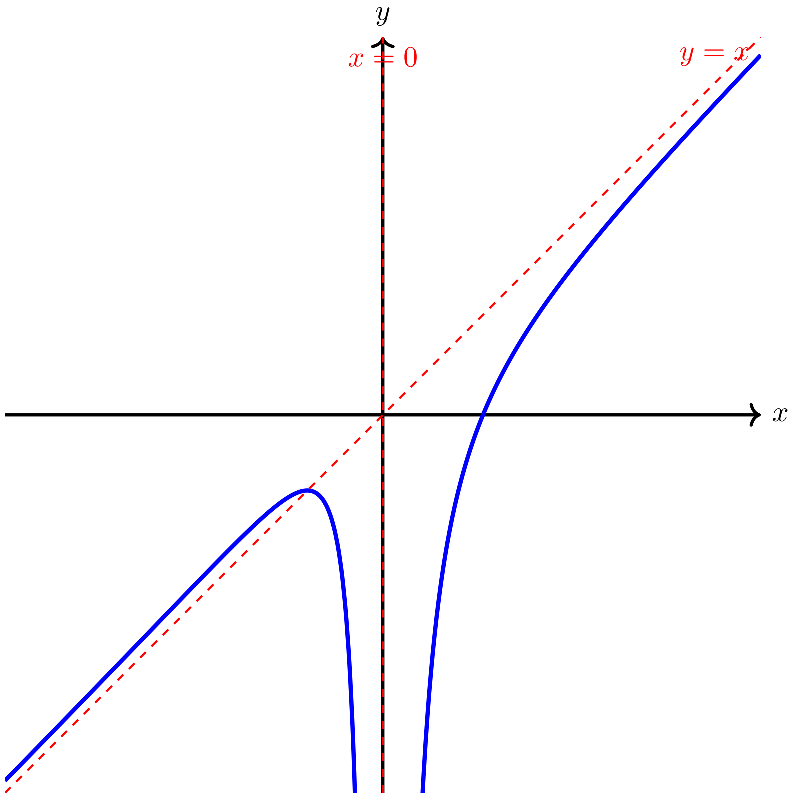
(iii) Given that $u+v+\dfrac{1}{uv} = -1$, and that u and v are real, show that $\dfrac{1}{u}+\dfrac{1}{v}+uv \leq -1$.

(iv) Given instead that $u+v+\dfrac{1}{uv} = 3$, and that u and v are real, find the greatest value of $\dfrac{1}{u} + \dfrac{1}{v} + uv$.

(i)

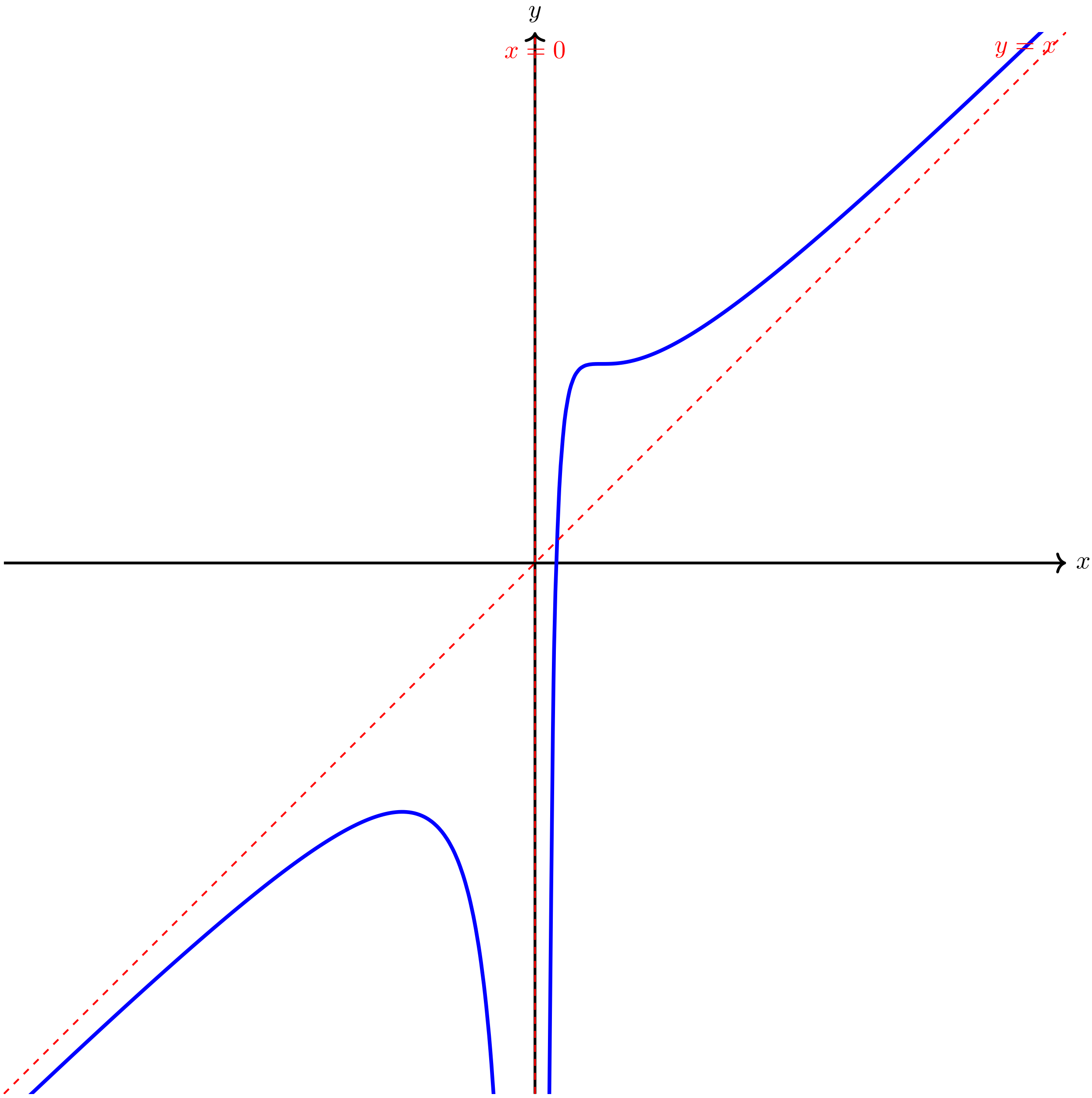
$$\begin{aligned}
&& f(\beta) &= \beta - \frac{1}{\beta} - \frac{1}{\beta^2} \\
\Rightarrow && f'(\beta) &= 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} \\
\Rightarrow && 0 &= f'(\beta) \\
&& &= 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} \\
\Rightarrow && 0 &= \beta^3 + \beta + 2 \\
&& &= (\beta+1)(\beta^2 - \beta + 2)
\end{aligned}$$

Therefore the only stationary point is at $\beta = -1, f(-1) = -1$



$$
\begin{aligned}
& & g(\beta) &= \beta + \frac{3}{\beta} - \frac{1}{\beta^2} \\
&\Rightarrow & g'(\beta) &= 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3} \\
&\Rightarrow & 0 &= f'(\beta) \\
& & &= 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3} \\
&\Rightarrow & 0 &= \beta^3 - 3\beta + 2 \\
& & &= (\beta - 1)^2(\beta + 2)
\end{aligned}
$$

Therefore there are stationary points at $\beta = 1, f(1) = 3, \beta = -2, f(-2) = \frac{1}{4}$



(ii) Let u, v be the roots of $x^2 + \alpha x + \beta = 0$, then since $(x-u)(x-v) = 0$ we must have $\alpha = -(u+v), \beta = uv$.

Therefore:

$$\begin{aligned} u + v + \frac{1}{uv} &= -\alpha + \frac{1}{\beta} \\ \frac{1}{u} + \frac{1}{v} + uv &= \frac{u+v}{uv} + uv \\ &= -\frac{\alpha}{\beta} + \beta \end{aligned}$$

Given $u + v + \frac{1}{uv} = -1$, ie $-\alpha + \frac{1}{\beta} = -1$. Since the roots are real, we must also have that $\alpha^2 - 4\beta \geq 0$, so

$$\begin{aligned}
& & -\alpha + \frac{1}{\beta} &= -1 \\
&\Rightarrow & \alpha &= 1 + \frac{1}{\beta} \\
&\Rightarrow & -\frac{\alpha}{\beta} + \beta &= -\frac{1}{\beta}\left(1 + \frac{1}{\beta}\right) + \beta \\
& & &= \beta - \frac{1}{\beta} - \frac{1}{\beta^2}
\end{aligned}$$

So we want to maximise $f(\beta)$ subject to $\alpha^2 - 4\beta \geq 0$

$$\begin{aligned}
& & 0 &\leq \alpha^2 - 4\beta \\
& & &= \left(1 + \frac{1}{\beta}\right)^2 - 4\beta \\
& & &= 1 + \frac{2}{\beta} + \frac{1}{\beta^2} - 4\beta \\
&\Leftrightarrow & 0 &\leq -4\beta^3 + \beta^2 + 2\beta + 1 \\
& & &= -(\beta - 1)(4\beta^2 + 3\beta + 1) \\
&\Leftrightarrow & \beta &\leq 1
\end{aligned}$$

But we know $f(\beta) \leq -1$ on $(-\infty, 1]$ so we're done.

(iii) Given that $-\alpha + \frac{1}{\beta} = 3$ we have

$$\begin{aligned}
& & -\alpha + \frac{1}{\beta} &= 3 \\
&\Rightarrow & \alpha &= -3 + \frac{1}{\beta} \\
&\Rightarrow & -\frac{\alpha}{\beta} + \beta &= -\frac{1}{\beta}\left(-3 + \frac{1}{\beta}\right) + \beta \\
& & &= \beta + \frac{3}{\beta} - \frac{1}{\beta^2}
\end{aligned}$$

which we want to maximise, subject to:

$$\begin{aligned}
& & 0 &\leq \alpha^2 - 4\beta \\
& & &= \left(-3 + \frac{1}{\beta}\right)^2 - 4\beta \\
& & &= 9 - \frac{6}{\beta} + \frac{1}{\beta^2} - 4\beta \\
&\Leftrightarrow & 0 &\leq -4\beta^3 + 9\beta^2 - 6\beta + 1 \\
& & &= -(\beta - 1)^2(4\beta - 1)
\end{aligned}$$

$$\Leftrightarrow \qquad \beta \leq \frac{1}{4}$$

Therefore the maximum will either be $f(-2) = \frac{1}{4}$ or $f(\frac{1}{4}) = -\frac{15}{4}$. Therefore the maximum is $\frac{1}{4}$

Question (2019 STEP III Q4)

The nth degree polynomial $\mathrm{P}(x)$ is said to be *reflexive* if:

(a) $\mathrm{P}(x)$ is of the form $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$ where $n \geq 1$;

(b) $a_1, a_2, \ldots, a_n$ are real;

(c) the n (not necessarily distinct) roots of the equation $\mathrm{P}(x) = 0$ are $a_1, a_2, \ldots, a_n$.

(i) Find all reflexive polynomials of degree less than or equal to 3.

(ii) For a reflexive polynomial with $n > 3$, show that

$$2a_2 = -a_2^2 - a_3^2 - \cdots - a_n^2.$$

Deduce that, if all the coefficients of a reflexive polynomial of degree n are integers and $a_n \neq 0$, then $n \leq 3$.

(iii) Determine all reflexive polynomials with integer coefficients.

(i) Suppose $n = 1$, then all polynomials are reflexive (since $x - a_1$ has the root a_1.

Suppose $n = 2$, then we want

$$\begin{aligned}
&& x^2 - a_1x + a_2 &= (x-a_1)(x-a_2) \\
&& &= x^2 - (a_1 + a_2)x + a_1a_2 \\
\Rightarrow && a_2 &= 0
\end{aligned}$$

So all polynomials of the form $x^2 - a_1x$ work and no others.

Suppose $n = 3$ then we want

$$\begin{aligned}
&& x^3 - a_1x^2 + a_2x - a_3 &= (x-a_1)(x-a_2)(x-a_3) \\
&& &= x^3 - (a_1+a_2+a_3)x + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 \\
\Rightarrow && a_2 + a_3 &= 0 \\
&& a_2a_3 &= a_2 \\
\Rightarrow && -a_2^2 &= a_2
\end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad a_2 &= 0, -1 \\ -a_1 a_2^2 &= -a_2 \\ \Rightarrow \qquad a_2 &= 0, a_2 = 1/a_1 \end{aligned}$$

So we need either $x^3 - a_1 x$ or $(x+1)^2(x-1) = x^3 + x^2 - x - 1$

(ii) Suppose $n > 3$ then

$$\begin{aligned} \sum a_i^2 &= \left(\sum a_i\right)^2 - 2\sum_{i<j} a_i a_j \\ &= a_1^2 - 2a_2 \\ \Rightarrow \qquad 2a_2 &= a_1^2 - \sum a_i^2 \\ &= -a_2^2 - a_3^2 - \cdots - a_n^2 \end{aligned}$$

So $(a_2+1)^2 = 1 - a_3^2 - \cdots - a_n^2$ so if $a_n > 0$ (or any other $a_i, i > 2$ for that matter) then we must have $a_n = \pm 1, a_3, \dots a_{n-1} = 0$, but if $a_n = \pm 1$ $x = 0$ is not a root. Therefore we must have a_0 and $a_i = 0$ for all $i > 3$

(iii) The only reflexive polynomials therefore must be $x^n - kx^{n-1}$ and $x^{n+3} + x^{n+2} - x^{n+1} - x^n$

Question (2025 STEP III Q6) **(i)** Let a, b and c be three non-zero complex numbers with the properties $a+b+c=0$ and $a^2+b^2+c^2=0$. Show that a, b and c cannot all be real. Show further that a, b and c all have the same modulus.

(ii) Show that it is not possible to find three non-zero complex numbers a, b and c with the properties $a+b+c=0$ and $a^3+b^3+c^3=0$.

(iii) Show that if any four non-zero complex numbers a, b, c and d have the properties $a+b+c+d=0$ and $a^3+b^3+c^3+d^3=0$, then at least two of them must have the same modulus.

(iv) Show, by taking $c=1$, $d=-2$ and $e=3$ that it is possible to find five real numbers a, b, c, d and e with distinct magnitudes and with the properties $a+b+c+d+e=0$ and $a^3+b^3+c^3+d^3+e^3=0$.

(i) If a,b,c were all real then $a^2+b^2+c^2=0 \Rightarrow a,b,c=0$ but they are non-zero. Therefore they cannot all be real.

Since $(a+b+c)^2 = 0$ we must have $ab+bc+ca=0$. Therefore a,b,c must satisfy $x^3 - abc = 0 \Rightarrow$ they all have the same modulus, since they are all cube roots of the same number.

(ii) Notice that $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \Rightarrow abc = 0$ but therefore they cannot all be non-zero.

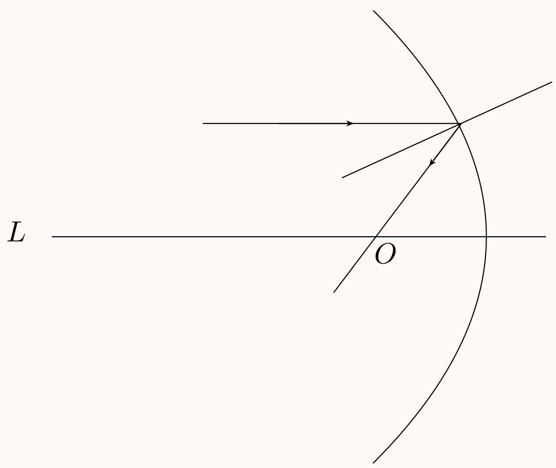
(iii) Suppose $a+b+c+d = 0$ then note that $a^2+b^2+c^2+d^2 = (a+b+c+d)^2 - 2\sum_{sym} ab$ and

$a^3+b^3+c^3+d^3 = (a+b+c+d)^3 - 3(a+b+c+d)(ab+ac+ad+bc+bd+cd) + 3(abc+abd+acd+bcd) \Rightarrow abc+abd+acd+bcd = 0$. Therefore a, b, c, d are roots of a polynomial of the form $x^4 - kx^2 + l = 0$, but this means they must come in pairs with the same modulus.

(iv) Suppose $c = 1, d = -2, e = 3$ so $c+d+e = 2$ and $c^3+d^3+e^3 = 1-8+27 = 20$, so we need to find a, b satisfying $a+b = -2, a^2+b^2 = -20$, ie $4 = (a+b)^2 = -20+2ab \Rightarrow ab = 12$, so we need the roots of $x^2+2x+12 = 0$ which clearly have different modulus.

Question (1989 STEP II Q5) **(i)** Show that in polar coordinates, the gradient of any curve at the point (r, θ) is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r\right) \Bigg/ \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta\right).$$



(ii) A mirror is designed so that any ray of light which hits one side of the mirror and which is parallel to a certain fixed line L is reflected through a fixed point O on L. For any ray hitting the mirror, the normal to the mirror at the point of reflection bisects the angle between the incident ray and the reflected ray, as shown in the figure. Prove that the mirror intersects any plane containing L in a parabola.

(i) Suppose our curve is $r(\theta)$, then $y = r\sin\theta, x = r\cos\theta$ and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$$

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta \\
\Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta}\bigg/\frac{\mathrm{d}x}{\mathrm{d}\theta} \\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r}{\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}
\end{aligned}
$$

as required.

(ii) Set up a system of polar coordinates such that the origin is at O and all points in the plane containing L are represented by (r, θ).

The constraint we have is that the angle of the normal, is $\frac{1}{2}\theta$. Let $\tan\frac{1}{2}\theta = t$, then $\tan\theta = \frac{2t}{1-t^2}$

$$
\begin{aligned}
\tan\frac{1}{2}\theta &= -\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r} \\
\Rightarrow \qquad t &= -\frac{r' - r\frac{2t}{1-t^2}}{r'\frac{2t}{1-t^2} + r} \\
&= \frac{2tr - (1-t^2)r'}{2tr' + (1-t^2)r} \\
\Rightarrow \qquad (2t^2 + 1 - t^2)r' &= (2t - t + t^3)r \\
(1+t^2)r' &= t(t^2+1)r \\
\Rightarrow \qquad r' &= tr \\
\Rightarrow \qquad \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \tan\tfrac{1}{2}\theta r \\
\Rightarrow \qquad \int \frac{1}{r}\mathrm{d}r &= \int \tan\frac{1}{2}\theta \mathrm{d}\theta \\
\ln r &= -2\ln\cos\tfrac{1}{2}\theta + C \\
\Rightarrow \qquad r\cos^2\frac{1}{2}\theta &= C \\
\Rightarrow \qquad r + r\cos\theta &= D \\
\Rightarrow \qquad r &= D - x \\
\Rightarrow \qquad x^2 + y^2 &= D^2 - 2Dx + x^2 \\
\Rightarrow \qquad y^2 &= D^2 - 2Dx
\end{aligned}
$$

Therefore it is a parabola

Question (1989 STEP III Q6)

Show that, for a given constant γ ($\sin\gamma \neq 0$) and with suitable choice of the constants A and B, the line with cartesian equation $lx + my = 1$ has polar equations

$$\frac{1}{r} = A\cos\theta + B\cos(\theta - \gamma).$$

The distinct points P and Q on the conic with polar equations

$$\frac{a}{r} = 1 + e\cos\theta$$

correspond to $\theta = \gamma - \delta$ and $\theta = \gamma + \delta$ respectively, and $\cos\delta \neq 0$. Obtain the polar equation of the chord PQ. Hence, or otherwise, obtain the equation of the tangent at the point where $\theta = \gamma$. The tangents at L and M to a conic with focus S meet at T. Show that ST bisects the angle LSM and find the position of the intersection of ST and LM in terms of your chosen parameters for L and M.

$$\begin{aligned}
\frac{1}{r} &= A\cos\theta + B\cos(\theta-\gamma)\\
&= A\cos\theta + B\cos\theta\cos\gamma + B\sin\theta\sin\gamma\\
&= (A + B\cos\gamma)\cos\theta + B\sin\gamma\sin\theta\\
\iff \quad 1 &= (A + B\cos\gamma)x + B\sin\gamma y
\end{aligned}$$

So if we choose $B = \frac{m}{\sin\gamma}$ and $A = l - m\cot\gamma$ we have the desired result.

$$\begin{aligned}
\frac{1 + e\cos(\gamma-\delta)}{a} &= A\cos(\gamma-\delta) + B\cos(\gamma-\delta-\gamma)\\
&= A\cos(\gamma-\delta) + B\cos\delta\\
\frac{1 + e\cos(\gamma+\delta)}{a} &= A\cos(\gamma+\delta) + B\cos(\gamma+\delta-\gamma)\\
&= A\cos(\gamma+\delta) + B\cos\delta\\
\Rightarrow \quad \frac{1}{r} &= \frac{e}{a}\cos\theta + \frac{1}{a\cos\delta}\cos(\theta-\gamma)\\
\lim\delta\to 0 \quad \frac{1}{r} &= \frac{e}{a}\cos\theta + \frac{1}{a}\cos(\theta-\gamma)
\end{aligned}$$

Suppose we have have points L and M with $\theta = \gamma_L, \gamma_M$ then our tangents are:

$$\begin{aligned}
\frac{a}{r} &= \cos\theta + \cos(\theta-\gamma_L)\\
\frac{a}{r} &= \cos\theta + \cos(\theta-\gamma_M)\\
\Rightarrow \quad 0 &= \cos(\theta-\gamma_L) - \cos(\theta-\gamma_M)\\
&= -2\sin\frac{(\theta-\gamma_L)+(\theta-\gamma_M)}{2}\sin\frac{(\theta-\gamma_L)-(\theta-\gamma_M)}{2}\\
&= -2\sin\left(\theta - \frac{\gamma_L+\gamma_M}{2}\right)\sin\left(\frac{\gamma_M-\gamma_L}{2}\right)\\
\Rightarrow \quad \theta &= \frac{\gamma_L+\gamma_M}{2}
\end{aligned}$$

Therefore clearly ST bisects LSM.
The line LM can be seen as the chord from the points $\frac{\gamma_L+\gamma_M}{2} \pm \frac{\gamma_L-\gamma_M}{2}$, so the line is:

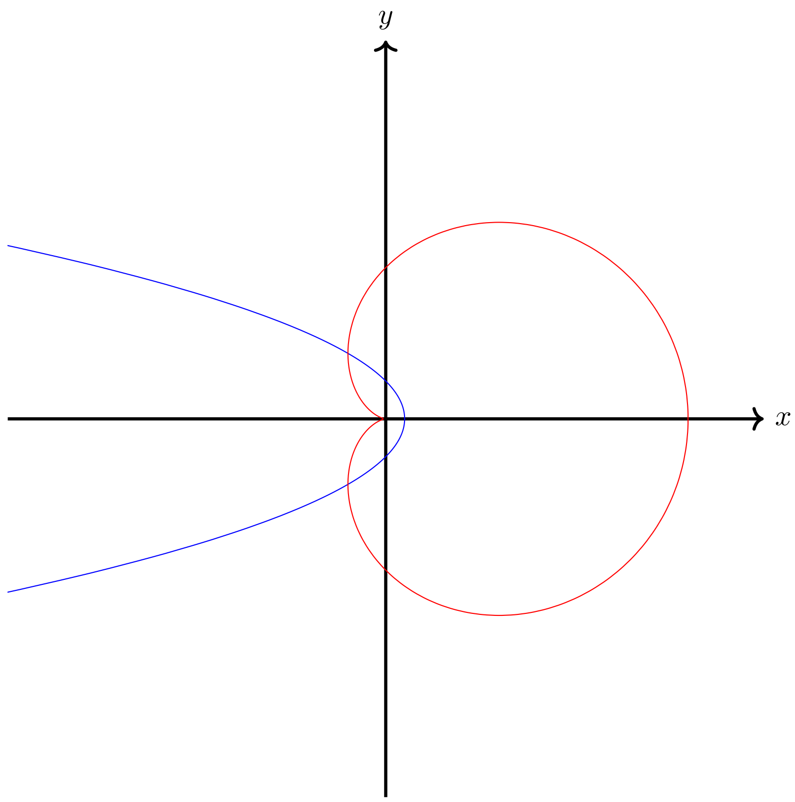
$$\frac{a}{r} = e\cos\theta + \frac{1}{\cos\left(\frac{\gamma_L-\gamma_M}{2}\right)}\cos\left(\theta - \frac{\gamma_L+\gamma_M}{2}\right)$$

and we want the point on the line where $\theta = \frac{\gamma_L+\gamma_M}{2}$ so

$$\begin{aligned} \frac{a}{r} &= e\cos\left(\frac{\gamma_L+\gamma_M}{2}\right) + \frac{1}{\cos\left(\frac{\gamma_L-\gamma_M}{2}\right)} \\ \Rightarrow \quad r &= \frac{a}{e\cos\left(\frac{\gamma_L+\gamma_M}{2}\right) + \frac{1}{\cos\left(\frac{\gamma_L-\gamma_M}{2}\right)}} \end{aligned}$$

Question (1990 STEP II Q9)

Show by means of a sketch that the parabola $r(1+\cos\theta) = 1$ cuts the interior of the cardioid $r = 4(1+\cos\theta)$ into two parts. Show that the total length of the boundary of the part that includes the point $r = 1, \theta = 0$ is $18\sqrt{3} + \ln(2+\sqrt{3})$.



The curves will intersect when:

$$\begin{aligned} \frac{1}{1+\cos\theta} &= 4(1+\cos\theta) \\ \Rightarrow \quad 1+\cos\theta &= \pm\frac{1}{2} \end{aligned}$$

$$\Rightarrow \qquad \cos\theta = -\frac{1}{2}$$
$$\Rightarrow \qquad \theta = \pm\frac{2\pi}{3},$$

Therefore we can measure the two sides of the boundaries. For the cardioid it will be:

$$\begin{aligned}
s &= \int_{-2\pi/3}^{2\pi/3} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta \\
&= \int_{-2\pi/3}^{2\pi/3} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta \\
&= \int_{-2\pi/3}^{2\pi/3} \sqrt{16(1+\cos\theta)^2 + 16\sin^2\theta\mathrm{d}\theta} \\
&= 4\int_{-2\pi/3}^{2\pi/3} \sqrt{2+2\cos\theta}\mathrm{d}\theta \\
&= 8\int_{-2\pi/3}^{2\pi/3} \sqrt{\cos^2\frac{\theta}{2}}\mathrm{d}\theta \\
&= 8\int_{-2\pi/3}^{2\pi/3} |\cos\frac{\theta}{2}|\mathrm{d}\theta \\
&= 16\int_{\pi}^{2\pi/3} (-\cos\frac{\theta}{2})\mathrm{d}\theta + 8\int_{-\pi}^{\pi}\cos\frac{\theta}{2}\mathrm{d}\theta \\
&= 16\cdot\left[2\sin\frac{\theta}{2}\right]_{\pi}^{2\pi/3} + 8\cdot 4 \\
&= 16\cdot(\sqrt{3}-2) + 8\cdot 4 \\
&= 16\sqrt{3}
\end{aligned}$$

For the parabola we have that $\sqrt{x^2+y^2}+x = 1 \Rightarrow x^2+y^2 = 1-2x+x^2 \Rightarrow y^2 = 1-2x$. So we can parameterise our parabola as $y = t, x = \frac{1-t^2}{2}$. And we are interested in the points $t = -\sqrt{3}$ and $t = \sqrt{3}$

$$\begin{aligned}
s &= \int_{-\sqrt{3}}^{\sqrt{3}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t} \\
&= \int_{-\sqrt{3}}^{\sqrt{3}\sqrt{t^2+1^2}\mathrm{d}t} \\
\sinh u = t, \frac{\mathrm{d}t}{\mathrm{d}u} = \cosh u \qquad &= \int_{-\sinh^{-1}\sqrt{3}}^{\sinh^{-1}\sqrt{3}} \cosh^2 u\mathrm{d}u \\
&= \left[\frac{1}{2}u + \frac{1}{4}\sinh(2u)\right]_{-\sinh^{-1}\sqrt{3}}^{\sinh^{-1}\sqrt{3}} \\
&= \sinh^{-1}\sqrt{3} + 2\sqrt{3} \\
&= \ln(2+\sqrt{3}) + 2\sqrt{3}
\end{aligned}$$

Therefore the total distance is as required.

Question (1991 STEP III Q5)

The curve C has the differential equation in polar coordinates

$$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta, \qquad \text{for} \quad \frac{\pi}{5} \leqslant \theta \leqslant \frac{3\pi}{5},$$

and, when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2$. Show that C forms a closed loop and that the area of the region enclosed by C is

$$\frac{\pi}{5} + \frac{25}{48}\left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\right].$$

First we seek the complementary function.

$$\begin{aligned} && \frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 4r = 0 \\ \Rightarrow && r = A\sin 2\theta + B\cos 2\theta \end{aligned}$$

Next we seek a particular integral, of the form $r = C\sin 3\theta$.

$$\begin{aligned} && \frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta \\ \Rightarrow && -9C\sin 3\theta + 4C\sin 3\theta = 5\sin 3\theta \\ \Rightarrow && C = -1 \end{aligned}$$

So our general solution is $A\sin 2\theta + B\cos 2\theta - \sin 3\theta$.
Plugging in boundary conditions we obtain:

$$\begin{aligned} \theta = \frac{\pi}{2}, r = 1: && 1 = -B + 1 \\ \Rightarrow && B = 0 \\ \theta = \frac{\pi}{2}, \frac{\mathrm{d}r}{\mathrm{d}\theta} = -2: && -2 = -2A \\ \Rightarrow && A = 1 \end{aligned}$$

So the general solution is $r = \sin 2\theta - \sin 3\theta = 2\sin\left(\frac{-\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right)$

First notice that for $\theta \in \left[\frac{\pi}{5}, \frac{3\pi}{5}\right]$ this is positive, and it is zero on the end points, therefore we are tracing out a a loop.

The area of the loop will be:

$$\begin{aligned} A &= \int_{\pi/5}^{3\pi/5} \frac{1}{2}\left(\sin 2\theta - \sin 3\theta\right)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\pi/5}^{3\pi/5} \sin^2 2\theta + \sin^2 3\theta - 2\sin 2\theta\cos 3\theta \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\pi/5}^{3\pi/5} \frac{1 - 2\cos 4\theta}{2} + \frac{1 - 2\cos 6\theta}{2} - \sin 5\theta - \cos\theta \mathrm{d}\theta \end{aligned}$$

$$= \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta - \frac{1}{6}\sin 6\theta + \frac{1}{5}\cos 5\theta - \sin\theta\right]_{\pi/5}^{3\pi/5}$$
$$= \frac{\pi}{5} + \frac{25}{48}\left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\right]$$

Question (1991 STEP III Q9)

The parametric equations E_1 and E_2 define the same ellipse, in terms of the parameters θ_1 and θ_2, (though not referred to the same coordinate axes).

$$E_1: \quad x = a\cos\theta_1, \qquad y = b\sin\theta_1,$$
$$E_2: \quad x = \frac{k\cos\theta_2}{1+e\cos\theta_2}, \quad y = \frac{k\sin\theta_2}{1+e\cos\theta_2},$$

where $0 < b < a$, $0 < e < 1$ and $0 < k$. Find the position of the axes for E_2 relative to the axes for E_1 and show that $k = a(1-e^2)$ and $b^2 = a^2(1-e^2)$. [The standard polar equation of an ellipse is $r = \dfrac{\ell}{1+e\cos\theta}$.] By considering expressions for the length of the perimeter of the ellipse, or otherwise, prove that

$$\int_0^\pi \sqrt{1-e^2\cos^2\theta}\,\mathrm{d}\theta = \int_0^\pi \frac{1-e^2}{(1+e\cos\theta)^2}\sqrt{1+e^2+2e\cos\theta}\,\mathrm{d}\theta.$$

Given that e is so small that e^6 may be neglected, show that the value of either integral is

$$\tfrac{1}{64}\pi(64 - 16e^2 - 3e^4).$$

None

Question (1992 STEP III Q10)

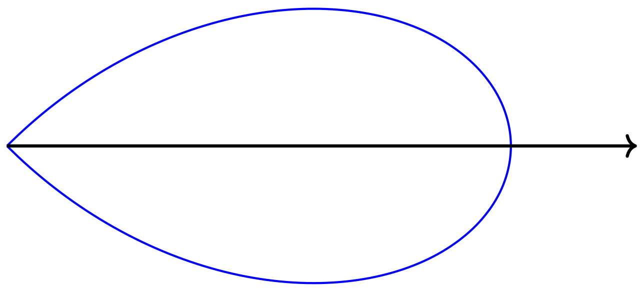
Sketch the curve C whose polar equation is

$$r = 4a\cos 2\theta \qquad \text{for } -\tfrac{1}{4}\pi < \theta < \tfrac{1}{4}\pi.$$

The ellipse E has parametric equations

$$x = 2a\cos\phi, \qquad y = a\sin\phi.$$

Show, without evaluating the integrals, that the perimeters of C and E are equal. Show also that the areas of the regions enclosed by C and E are equal.



$$\begin{aligned}\text{Perimeter}(C) &= \int_{-\pi/4}^{\pi/4} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta \\ &= \int_{-\pi/4}^{\pi/4} \sqrt{16a^2\cos^2 2\theta + 64a^2\sin^2 2\theta}\mathrm{d}\theta \\ &= \int_{-\pi/4}^{\pi/4} 4a\sqrt{1+3\sin^2 2\theta}\mathrm{d}\theta\end{aligned}$$

$$\begin{aligned}\text{Perimeter}(D) &= \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\phi}\right)^2}\mathrm{d}\phi \\ &= \int_0^{2\pi} \sqrt{4a^2\sin^2\phi + a^2\cos^2\phi\mathrm{d}\phi} \\ &= a^2\int_0^{2\pi} \sqrt{1+3\sin^2\phi}\mathrm{d}\phi\end{aligned}$$

But clearly these two integrals are equal.

$$\begin{aligned}\mathrm{A}(C) &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\mathrm{d}\theta \\ &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} 16a^2\cos^2 2\theta\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&= 8a^2 \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \mathrm{d}\theta \\
&= 8a^2 \frac{\pi}{4} = 2\pi a^2 \\
\mathrm{A}(D) &= 2\pi a^2
\end{aligned}
$$

Question (1993 STEP II Q5)

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0,linewidth=0.5pt,arrowsize=3pt 2,arrowinset=0.25 (-0.57,-0.63)(8.51,6.23)
(0,0)(7,5) (7,5)(7.75,1.98) (7.75,1.98)(0,0) -
0.67408182176363680.222081901905479941*5.52*cos(t)+0*5.52*sin(t)+1.48—0*5.52*cos(t)+1*5.52*sin(t)-
(7,5)(5.79,1.45) [tl](-0.4,-0.02)O [tl](5.76,1.29)P [tl](8.1,2.01)R [tl](7.2,5.26)Q
(7.67,2.29)(7.37,2.22) (7.37,2.22)(7.45,1.91)

In the diagram, O is the origin, P is a point of a curve $r = r(\theta)$ with coordinates (r, θ) and Q is another point of the curve, close to P, with coordinates $(r+\delta r, \theta+\delta\theta)$. The angle $\angle PRQ$ is a right angle. By calculating $\tan\angle QPR$, show that the angle at which the curve cuts OP is

$$\tan^{-1}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right).$$

Let α be a constant angle, $0 < \alpha < \frac{1}{2}\pi$. The curve with the equation

$$r = \mathrm{e}^{\theta\cot\alpha}$$

in polar coordinates is called an *equiangular spiral*. Show that it cuts every radius line at an angle α. Sketch the spiral.

Find the length of the complete turn of the spiral beginning at $r = 1$ and going outwards. What is the total length of the part of the spiral for which $r \leqslant 1$?

[You may assume that the arc length s of the curve satisfies

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2.]$$

Question (1993 STEP III Q2)

The curve C has the equation $x^3 + y^3 = 3xy$.

(i) Show that there is no point of inflection on C. You may assume that the origin is not a point of inflection.

(ii) The part of C which lies in the first quadrant is a closed loop touching the axes at the origin. By converting to polar coordinates, or otherwise, evaluate the area of this loop.

Question (1998 STEP III Q4)

Show that the equation (in plane polar coordinates) $r = \cos\theta$, for $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$, represents a circle. Sketch the curve $r = \cos 2\theta$ for $0 \leq \theta \leq 2\pi$, and describe the curves $r = \cos 2n\theta$, where n is an integer. Show that the area enclosed by such a curve is independent of n. Sketch also the curve $r = \cos 3\theta$ for $0 \leq \theta \leq 2\pi$.

Question (2006 STEP III Q6)

Show that in polar coordinates the gradient of any curve at the point (r, θ) is

$$\frac{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\tan\theta + r}{\dfrac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta}\,.$$

xunit=1.0cm,yunit=1.0cm,algebraic=true,dotstyle=o,dotsize=3pt 0,linewidth=0.5pt,arrowsize=3pt 2,arrowinset=0.25 (-0.6,-3)(6.8,3) (0,0)(6.54,0) [tl](4.13,-0.22)O [tl](-0.47,0.07)L -270(5.75,0.08)[plotpoints=500]-1212x^2/2/3$(2,1.5)(5.42,1.5)(3.73,-0.74)(5.42,1.5)[linewidth=0.4pt]->(3,1.5)(4,1.5)[linewidth=0.4pt]->(5.42,1.5)(4.99,0.93)(3.84,0.78)(6.62,2.05)$

A mirror is designed so that if an incident ray of light is parallel to a fixed line L the reflected ray passes through a fixed point O on L. Prove that the mirror intersects any plane containing L in a parabola. You should assume that the angle between the incident ray and the normal to the mirror is the same as the angle between the reflected ray and the normal.

Question (2011 STEP III Q5)

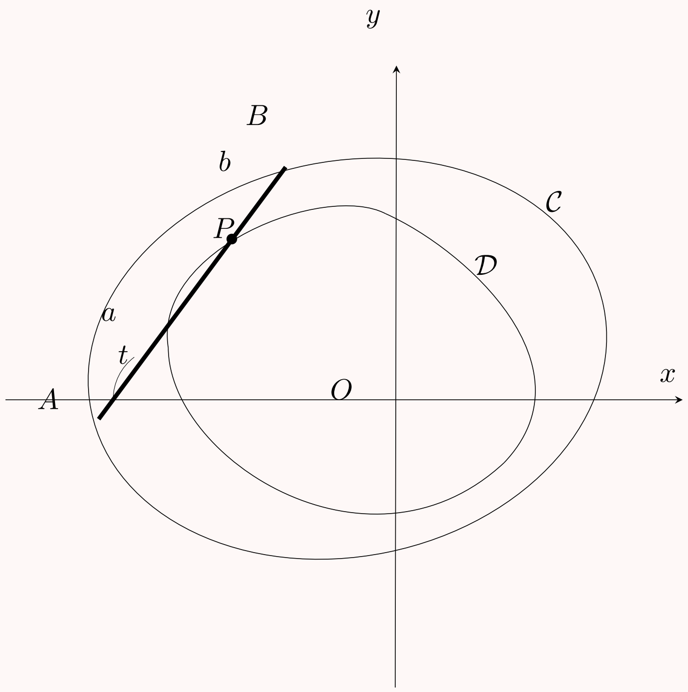
A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2 \, \mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t\,. \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t. The areas swept out by OA, OB and OP are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi} \left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t} \right) \cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t} \right) \sin t \right) \mathrm{d}t\,.$$

Obtain a corresponding expression for $[B]$ involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

$$\tan\theta = y/x$$

$$\Rightarrow \qquad \sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{x^2}$$

$$\Rightarrow \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\frac{\cos^2\theta}{x^2}$$

$$= \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\frac{\cos^2\theta}{r^2\cos^2\theta}$$

$$= \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\frac{1}{r^2}$$

$$\tfrac{1}{2}\int r^2\,\mathrm{d}\theta = \tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

$A = (x - a\cos t, y - a\sin t), B = (x + b\cos t, y + b\sin t)$

$$\begin{aligned}
[A] &= \tfrac{1}{2}\int_0^{2\pi}\left((x - a\cos t)\frac{\mathrm{d}(y - a\sin t)}{\mathrm{d}t} - (y - a\sin t)\frac{\mathrm{d}(x - a\cos t)}{\mathrm{d}t}\right)\mathrm{d}t \\
&= \tfrac{1}{2}\int_0^{2\pi}\left((x - a\cos t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} - a\cos t\right) - (y - a\sin t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + a\sin t\right)\right)\mathrm{d}t \\
&= \tfrac{1}{2}\int_0^{2\pi}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} - a\cos t\frac{\mathrm{d}y}{\mathrm{d}t} - ax\cos t + a^2\cos^2 t + a\sin t\frac{\mathrm{d}x}{\mathrm{d}t} - ya\sin t + a^2\sin^2 t\right)\mathrm{d}t \\
&= \tfrac{1}{2}\int_0^{2\pi}\left(\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}_{[P]} - a\left((x + \frac{\mathrm{d}y}{\mathrm{d}x})\cos t + (y - \frac{\mathrm{d}x}{\mathrm{d}t})\sin t\right) + \underbrace{a^2}_{\pi a^2}\right)\mathrm{d}t \\
&= [P] + \pi a^2 - af
\end{aligned}$$

$$\begin{aligned}
[B] &= \tfrac{1}{2}\int_0^{2\pi}\left((x + b\cos t)\frac{\mathrm{d}(y + b\sin t)}{\mathrm{d}t} - (y + b\sin t)\frac{\mathrm{d}(x + b\cos t)}{\mathrm{d}t}\right)\mathrm{d}t \\
&= \tfrac{1}{2}\int_0^{2\pi}\left((x + b\cos t)(\frac{\mathrm{d}y}{\mathrm{d}t} + b\cos t) - (y + b\sin t)(\frac{\mathrm{d}x}{\mathrm{d}t} - b\sin t)\right)\mathrm{d}t \\
&= \tfrac{1}{2}\int_0^{2\pi}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} + b^2 + b(\cos t(x + \frac{\mathrm{d}y}{\mathrm{d}t}) + (y - \frac{\mathrm{d}x}{\mathrm{d}t})\sin t\right)\mathrm{d}t \\
&= [P] + \pi b^2 + bf
\end{aligned}$$

Since A and B trace out the same area, we must have $\pi a^2 - af = \pi b^2 + bf \Rightarrow \pi(a^2 - b^2) = f(b + a) \Rightarrow f = \pi(a - b)$.
In particular the area inbetween is $[A] - [P] = \pi a^2 - a\pi(a - b)$

Question (2015 STEP III Q3)

In this question, r and θ are polar coordinates with $r \geq 0$ and $-\pi < \theta \leq \pi$, and a and b are positive constants. Let L be a fixed line and let A be a fixed point not lying on L. Then the locus of points that are a fixed distance (call it d) from L measured along lines through A is called a *conchoid of Nicomedes*.

(i) Show that if

$$|r - a\sec\theta| = b\,, \qquad (*)$$

where $a > b$, then $\sec\theta > 0$. Show that all points with coordinates satisfying $(*)$ lie on a certain conchoid of Nicomedes (you should identify L, d and A). Sketch the locus of these points.

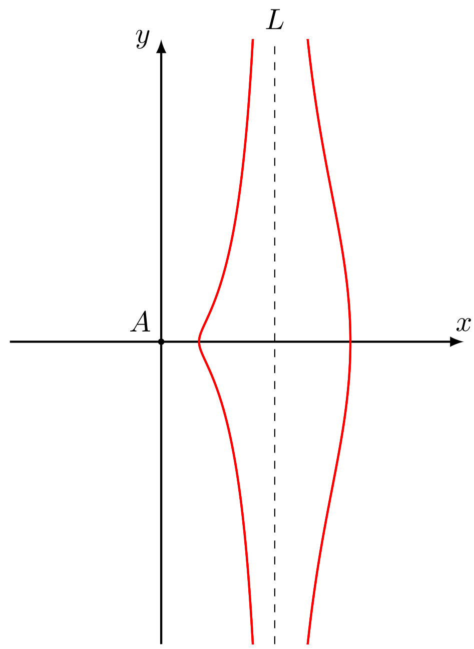
(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec\theta < 0$) given by

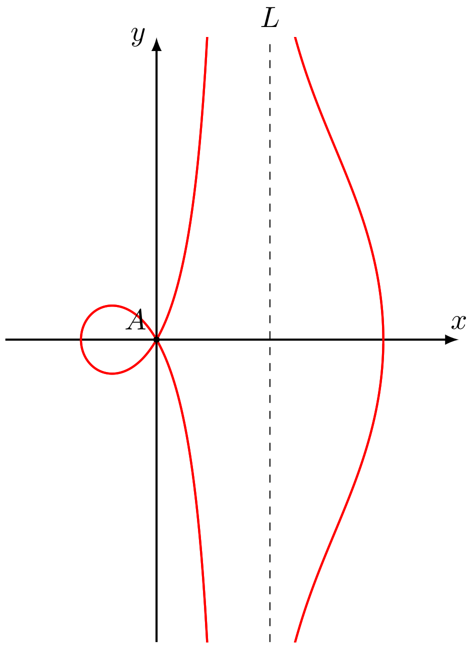
$$|r - a\sec\theta| = b\,.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[Note: $\int \sec\theta \,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.]

(i) $r = a\sec\theta \pm b$. The points on $r = a\sec\theta \Leftrightarrow r\cos\theta = a \Leftrightarrow x = a$ are points on the line $x = a$. Therefore points on the curve $r = a\sec\theta \pm b$ are points which are a distance b from the line $x = a$ measured towards O. So A is the origin and $d = b$.





(ii)

The loop starts and ends when $r = a\sec\theta - b = 0 \Rightarrow \cos\theta = \frac{a}{b}$, so when $a = 1, b = 2$, this is $-\frac{\pi}{3}$ to $\frac{\pi}{3}$

$$
\begin{aligned}
A &= \frac{1}{2}\int r^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (\sec\theta - 2)^2 \,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(\sec^2\theta - 4\sec\theta + 4\right) \mathrm{d}\theta \\
&= \frac{1}{2}\left[\tan\theta - 4\ln|\sec\theta + \tan\theta| + 4\theta\right]_{-\pi/3}^{\pi/3} \\
&= \frac{1}{2}\left(\left(\tan\frac{\pi}{3} - 4\ln|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}| + 4\left(\frac{\pi}{3}\right)\right) - \left(\tan\left(-\frac{\pi}{3}\right) - 4\ln|\sec\left(-\frac{\pi}{3}\right) + \tan\left(-\frac{\pi}{3}\right)| + 4\left(\right.\right.\right. \\
&= \frac{1}{2}\left(2\sqrt{3} - 4\ln|2+\sqrt{3}| + 4\ln|2-\sqrt{3}| + \frac{8\pi}{3}\right) \\
&= \sqrt{3} + 2\ln\frac{2-\sqrt{3}}{2+\sqrt{3}} + \frac{4\pi}{3} \\
&= \sqrt{3} + 4\ln(2-\sqrt{3}) + \frac{4\pi}{3}
\end{aligned}
$$

Question (2015 STEP III Q8) **(i)** Show that under the changes of variable $x = r\cos\theta$ and $y = r\sin\theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y+x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$

becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0\,.$$

Sketch a solution in the x-y plane.

(ii) Show that the solutions of

$$\big(y + x - x(x^2+y^2)\big)\,\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - y(x^2+y^2)$$

can be written in the form

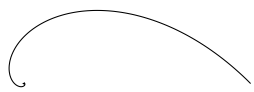
$$r^2 = \frac{1}{1 + A\mathrm{e}^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A.

(i)

$$\begin{aligned}
& & (y+x)\frac{\mathrm{d}y}{\mathrm{d}x} &= y - x \\
&\Rightarrow & (r\sin\theta + r\cos\theta)\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} &= (r\sin\theta - r\cos\theta) \\
&\Rightarrow & (\sin\theta + \cos\theta)\frac{dy}{d\theta} &= (\sin\theta - \cos\theta)\frac{dx}{d\theta} \\
&\Rightarrow & (\sin\theta + \cos\theta)\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) &= (\sin\theta - \cos\theta)\left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) \\
&\Rightarrow & \frac{dr}{d\theta}\left(\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta + \sin\theta\cos\theta\right) &= r\left(\sin\theta\cos\theta - \cos^2\theta + \sin^2\theta + \sin\theta\cos\theta\right) \\
&\Rightarrow & \frac{dr}{d\theta} &= -r
\end{aligned}$$

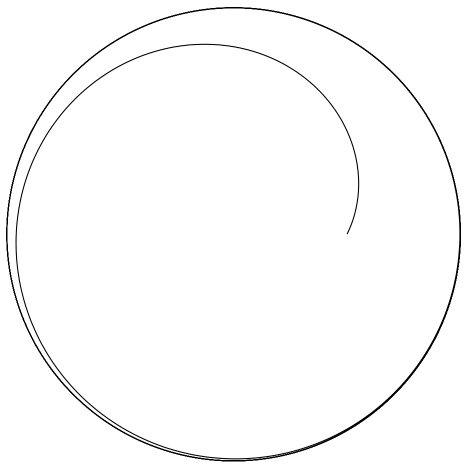
Therefore $r = Ae^{-\theta}$

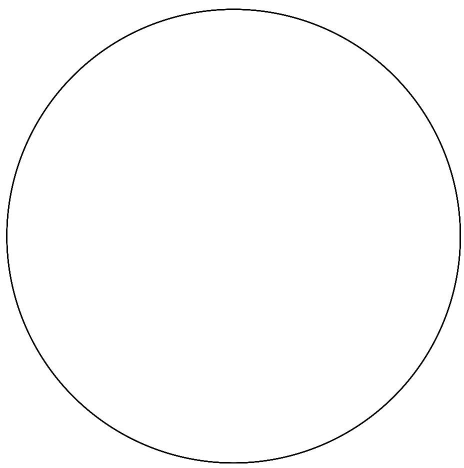


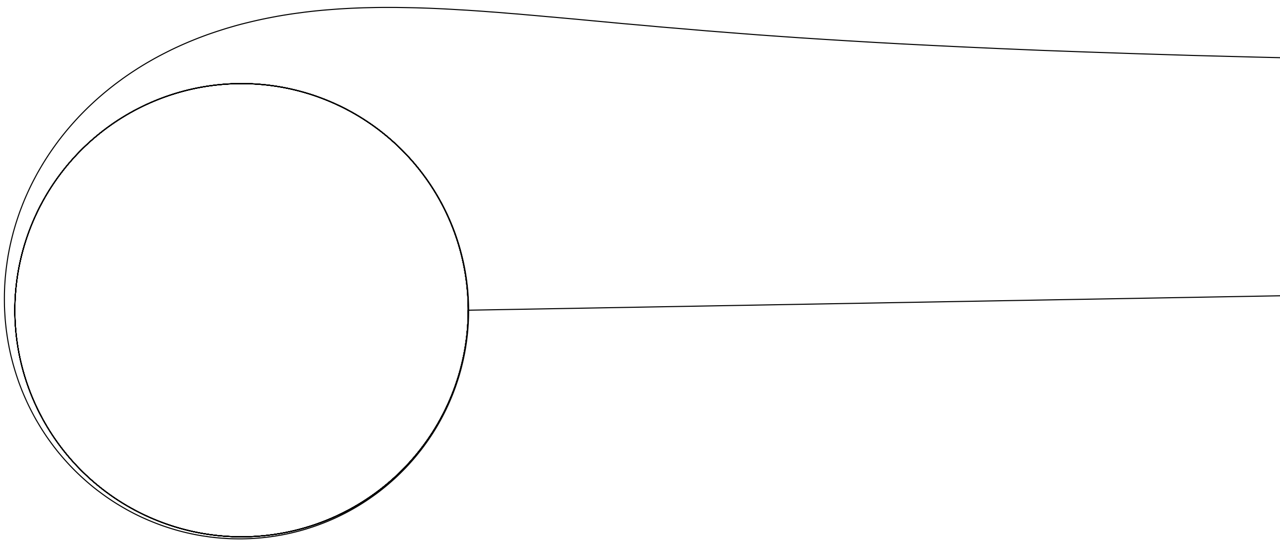
(ii)

$$\begin{aligned}
& & \big(y + x - x(x^2+y^2)\big)\,\frac{\mathrm{d}y}{\mathrm{d}x} &= y - x - y(x^2+y^2) \\
&\Rightarrow & \big(r\sin\theta + r\cos\theta - r^3\cos\theta\big)\,\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \big(r\sin\theta - r\cos\theta - r^3\sin\theta\big)\,\frac{\mathrm{d}x}{\mathrm{d}\theta}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & \left(r\sin\theta + r\cos\theta - r^3\cos\theta\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right) = \\
& \qquad \left(r\sin\theta - r\cos\theta - r^3\sin\theta\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right) \\
\Rightarrow \quad & \frac{\mathrm{d}r}{\mathrm{d}\theta}\left(\sin\theta(\sin\theta + \cos\theta - r^2\cos\theta) - \cos\theta(\sin\theta - \cos\theta - r^2\sin\theta)\right) = \\
& \qquad r(-\sin\theta(\sin\theta - \cos\theta - r^2\sin\theta) - \cos\theta(\sin\theta + \cos\theta - r^2\cos\theta)) \\
\Rightarrow \quad & \frac{\mathrm{d}r}{\mathrm{d}\theta} = r(-1 + r^2) \\
\Rightarrow \quad & \int \frac{1}{r(r-1)(r+1)}\mathrm{d}r = \int \mathrm{d}\theta \\
\Rightarrow \quad & \int \left(\frac{-1}{r} + \frac{1}{2(r-1)} + \frac{1}{2(r+1)}\right)\mathrm{d}r = \int \mathrm{d}\theta \\
\Rightarrow \quad & \left(-\log r + \frac{1}{2}\log(1+r) + \frac{1}{2}\log(1-r)\right) + C = \theta \\
\Rightarrow \quad & \frac{1}{2}\log\left(\frac{1-r^2}{r^2}\right) + C = \theta \\
\Rightarrow \quad & \log\left(\frac{1}{r^2} - 1\right) + C = 2\theta \\
\Rightarrow \quad & r = \frac{1}{1 + Ae^{2\theta}}
\end{aligned}$$







Question (2017 STEP III Q5)

The point with cartesian coordinates (x, y) lies on a curve with polar equation $r = f(\theta)$. Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $f(\theta)$, $f'(\theta)$ and $\tan\theta$.

Two curves, with polar equations $r = f(\theta)$ and $r = g(\theta)$, meet at right angles. Show that where they meet

$$f'(\theta)g'(\theta) + f(\theta)g(\theta) = 0\,.$$

The curve C has polar equation $r = f(\theta)$ and passes through the point given by $r = 4$, $\theta = -\frac{1}{2}\pi$. For each positive value of a, the curve with polar equation $r = a(1 + \sin\theta)$ meets C at right angles. Find $f(\theta)$.

Sketch on a single diagram the three curves with polar equations $r = 1 + \sin\theta$, $r = 4(1 + \sin\theta)$ and $r = f(\theta)$.

$(x, y) = (f(\theta)\cos(\theta), f(\theta)\sin(\theta))$ so

$$\begin{aligned}
\frac{dy}{d\theta} &= -f(\theta)\sin(\theta) + f'(\theta)\cos(\theta) \\
\frac{dx}{d\theta} &= f(\theta)\cos(\theta) + f'(\theta)\sin(\theta) \\
\frac{dy}{dx} &= \frac{-f(\theta)\sin(\theta) + f'(\theta)\cos(\theta)}{f(\theta)\cos(\theta) + f'(\theta)\sin(\theta)} \\
&= \frac{-f(\theta)\tan(\theta) + f'(\theta)}{f(\theta) + f'(\theta)\tan(\theta)}
\end{aligned}$$

If the curves meet at right angles then the product of their gradients is -1, ie

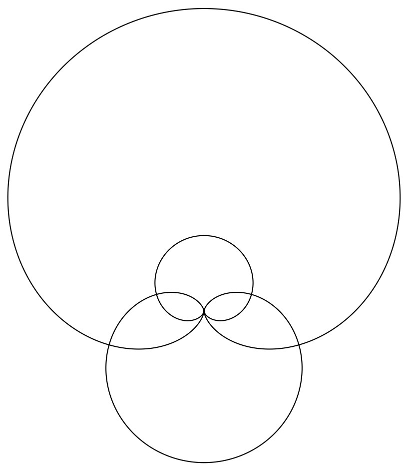
$$\begin{aligned}
\frac{-f(\theta)\tan(\theta) + f'(\theta)}{f(\theta) + f'(\theta)\tan(\theta)} \cdot \frac{-g(\theta)\tan(\theta) + g'(\theta)}{g(\theta) + g'(\theta)\tan(\theta)} &= -1 \\
f(\theta)g(\theta)\tan^2\theta - f(\theta)g'(\theta)\tan\theta - f'(\theta)g(\theta)\tan\theta + f'(\theta)g'(\theta) &= \\
-\left(f(\theta)g(\theta) + f(\theta)g'(\theta)\tan(\theta) + f'(\theta)g(\theta)\tan(\theta) + f'(\theta)g'(\theta)\tan^2\theta\right) \\
\tan^2\theta\left(f(\theta)g(\theta) + f'(\theta)g'(\theta)\right) + f'(\theta)g'(\theta) + f(\theta)g(\theta) &= 0
\end{aligned}$$

$$
\begin{aligned}
(\tan^2\theta + 1)\left(f(\theta)g(\theta) + f'(\theta)g'(\theta)\right) &= 0 \\
f(\theta)g(\theta) + f'(\theta)g'(\theta) &= 0
\end{aligned}
$$

$g(\theta) = a(1 + \sin\theta), g'(\theta) = a\cos\theta$
Therefore $f'(\theta)a\cos\theta + f(\theta)a(1 + \sin(\theta)) = 0$

$$
\begin{aligned}
\frac{f'(\theta)}{f(\theta)} &= -\sec(\theta) - \tan(\theta) \\
\Rightarrow \quad \ln(f(\theta)) &= -\ln|\tan(\theta) + \sec(\theta)| + \ln|\cos(\theta)| + C \\
\Rightarrow \quad f(\theta) &= A\frac{\cos\theta}{\tan\theta + \sec\theta} \\
&= A\frac{\cos^2\theta}{\sin\theta + 1} \\
&= A\frac{1 - \sin^2\theta}{\sin\theta + 1} \\
&= A(1 - \sin\theta)
\end{aligned}
$$

When $\theta = -\frac{1}{2}\pi, r = 4$, so $A = 2$.



Question (2018 STEP III Q4)

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

where $a > b > 0$. Show that the equation of the tangent to the hyperbola at P can be written as

$$bx - ay\sin\theta = ab\cos\theta\,.$$

(i) This tangent meets the lines $\frac{x}{a} = \frac{y}{b}$ and $\frac{x}{a} = -\frac{y}{b}$ at S and T, respectively. How is the mid-point of ST related to P?

(ii) The point $Q(a\sec\phi, b\tan\phi)$ also lies on the hyperbola and the tangents to the hyperbola at P and Q are perpendicular. These two tangents intersect at (x, y). Obtain expressions for x^2 and y^2 in terms of a, θ and ϕ. Hence, or otherwise, show that $x^2 + y^2 = a^2 - b^2$.

Note that

$$\begin{aligned}
\frac{\mathrm{d}a\sec\theta}{\mathrm{d}\theta} &= a\sec\theta\tan\theta \\
\frac{\mathrm{d}b\tan\theta}{\mathrm{d}\theta} &= b\sec^2\theta \\
\Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{b\sec^2\theta}{a\sec\theta\tan\theta} \\
&= \frac{b}{a}\frac{1}{\sin\theta} \\
\Rightarrow \qquad \frac{y - b\tan\theta}{x - a\sec\theta} &= \frac{b}{a}\frac{1}{\sin\theta} \\
\Rightarrow \qquad a\sin\theta y - ab\tan\theta\sin\theta &= bx - ab\sec\theta \\
\Rightarrow \qquad bx - ay\sin\theta &= ab\sec x(1 - \sin^2\theta) \\
&= ab\cos\theta
\end{aligned}$$

(i)

$$\begin{aligned}
S: \qquad & \begin{cases} bx - ay & = 0 \\ bx - ay\sin\theta & = ab\cos\theta \end{cases} \\
\Rightarrow \qquad ay(1 - \sin\theta) &= ab\cos\theta \\
\Rightarrow \qquad y &= \frac{b\cos\theta}{1 - \sin\theta} \\
x &= \frac{a\cos\theta}{1 - \sin\theta} \\
T: \qquad & \begin{cases} bx + ay & = 0 \\ bx - ay\sin\theta & = ab\cos\theta \end{cases} \\
\Rightarrow \qquad ay(1 + \sin\theta) &= -ab\cos\theta \\
\Rightarrow \qquad y &= \frac{-b\cos\theta}{1 + \sin\theta} \\
x &= \frac{a\cos\theta}{1 + \sin\theta}
\end{aligned}$$

$$M: \quad \begin{aligned} x &= \frac{a\cos\theta}{2}\frac{2}{1-\sin^2\theta} \\ &= a\sec\theta \\ y &= \frac{b\cos\theta}{2}\frac{2\sin\theta}{1-\sin^2\theta} \\ &= b\tan\theta \end{aligned}$$

The midpoint of ST is the same as P.

(ii) The tangents are perpendicular, therefore $\frac{b}{a}\theta = -\frac{a}{b}\sin\phi$, ie $b^2 = -a^2\sin\phi\sin\theta$

The will intersect at:

$$\begin{cases} bx - ay\sin\theta &= ab\cos\theta \\ bx - ay\sin\phi &= ab\cos\phi \end{cases}$$

$$\begin{aligned} \Rightarrow \quad ay(\sin\theta - \sin\phi) &= ab(\cos\phi - \cos\theta) \\ \Rightarrow \quad y &= \frac{b(\cos\phi - \cos\theta)}{(\sin\theta - \sin\phi)} \\ y^2 &= \frac{-a^2\sin\phi\sin\theta(\cos\phi - \cos\theta)^2}{(\sin\theta - \sin\phi)^2} \\ \Rightarrow \quad bx(\sin\phi - \sin\theta) &= ab(\cos\theta\sin\phi - \cos\phi\sin\theta) \\ \Rightarrow \quad x &= \frac{a(\cos\theta\sin\phi - \cos\phi\sin\theta)}{\sin\phi - \sin\theta} \\ &= \frac{a^2(\cos\theta\sin\phi - \cos\phi\sin\theta)^2}{(\sin\phi - \sin\theta)^2} \end{aligned}$$

Therefore

$$\begin{aligned} x^2 + y^2 &= \frac{a^2}{(\sin\phi - \sin\theta)^2}\left((\cos\theta\sin\phi - \cos\phi\sin\theta)^2 - \sin\phi\sin\theta(\cos\phi - \cos\theta)^2\right) \\ &= \frac{a^2}{(\sin\phi - \sin\theta)^2}\left((\sin\phi - \sin\theta)(\cos^2\theta\sin\phi - \sin\theta\cos^2\phi)\right) \\ &= a^2 - b^2 \end{aligned}$$

Question (1988 STEP III Q8)

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. For $i = 1, 2,$ and 3, let P_i be the point $(at_i^2, 2at_i)$, where t_1, t_2 and t_3 are all distinct. Let A_1 be the area of the triangle formed by the tangents at P_1, P_2 and P_3, and let A_2 be the area of the triangle formed by the normals at P_1, P_2 and P_3. Using the fact that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of

$$\tfrac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix},$$

show that $A_3 = (t_1 + t_2 + t_3)^2 A_1$. Deduce a necessary and sufficient condition in terms of t_1, t_2 and t_3 for the normals at P_1, P_2 and P_3 to be concurrent.

$\frac{dy}{dt} = 2a, \frac{dx}{dt} = 2at \Rightarrow \frac{dy}{dx} = \frac{1}{t}$.

Therefore the equation of the tangent will be $\frac{y-2at}{x-at^2} = \frac{1}{t} \Rightarrow y = \frac{1}{t}x + at$ and normal will be $\frac{y-2at}{x-at^2} = -t \Rightarrow y = t(at^2 - x + 2a)$.

The tangents will meet when:

$$\begin{cases} t_i y - x &= at_i^2 \\ t_j y - x &= at_j^2 \end{cases}$$

$$\begin{aligned} \Rightarrow \quad (t_i - t_j)y &= a(t_i - t_j)(t_i + t_j) \\ \Rightarrow \quad y &= a(t_i + t_j) \\ x &= at_i t_j \end{aligned}$$

The normals will meet when:

$$\begin{cases} y + t_i x &= at_i^3 + 2at_i \\ y + t_j x &= at_j^3 + 2at_j \end{cases}$$

$$\begin{aligned} \Rightarrow \quad (t_i - t_j)x &= a(t_i - t_j)(t_i^2 + t_i t_j + t_j^2 + 2) \\ \Rightarrow \quad x &= a(t_i^2 + t_i t_j + t_j^2 + 2) \\ y &= -at_i t_j (t_i + t_j) \end{aligned}$$

Therefore the area of our triangles will be:

$$\begin{aligned} \tfrac{1}{2}\det\begin{pmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \end{pmatrix} &= \frac{a^2}{2}\det\begin{pmatrix} t_1t_2 & (t_1+t_2) & 1 \\ t_2t_3 & (t_2+t_3) & 1 \\ t_3t_1 & (t_3+t_1) & 1 \end{pmatrix} \\ &= \frac{a^2}{2}\det\begin{pmatrix} t_1t_2 & (t_1+t_2) & 1 \\ t_2(t_3-t_1) & (t_3-t_1) & 0 \\ t_1(t_3-t_2) & (t_3-t_2) & 0 \end{pmatrix} \\ &= \frac{a^2}{2}|(t_2-t_1)(t_3-t_2)(t_1-t_3)| \end{aligned}$$

and

$$\tfrac{1}{2}\det\begin{pmatrix} a(t_1^2+t_1t_2+t_2^2+2) & -at_1t_2(t_1+t_2) & 1 \\ a(t_2^2+t_2t_3+t_3^2+2) & -at_2t_3(t_2+t_3) & 1 \\ a(t_3^2+t_3t_1+t_1^2+2) & -at_3t_1(t_3+t_1) & 1 \end{pmatrix} = \frac{a^2}{2}\det\begin{pmatrix} (t_1^2+t_1t_2+t_2^2+2) & -t_1t_2(t_1+t_2) & 1 \\ (t_2^2+t_2t_3+t_3^2+2) & -t_2t_3(t_2+t_3) & 1 \\ (t_3^2+t_3t_1+t_1^2+2) & -t_3t_1(t_3+t_1) & 1 \end{pmatrix}$$

$$= \frac{a^2}{2} \det \begin{pmatrix} (t_1^2+t_1t_2+t_2^2+2) & -t_1t_2(t_1+t_2) \\ t_3^2-t_1^2+t_2(t_3-t_1) & t_2(t_1^2+t_1t_2-t_2t_3-t_3^2) \\ t_3^2-t_2^2+t_1(t_3-t_2) & t_1(t_2^2+t_2t_1-t_1t_3-t_3^2) \end{pmatrix}$$

$$= \frac{a^2}{2} \det \begin{pmatrix} (t_1^2+t_1t_2+t_2^2+2) & -t_1t_2(t_1+t_2) \\ (t_3-t_1)(t_3+t_2+t_1) & t_2(t_1-t_3)(t_1+t_3+t_2 \\ (t_3-t_2)(t_3+t_2+t_1) & t_1(t_2-t_3)(t_1+t_2+t_3 \end{pmatrix}$$

$$= \frac{a^2}{2}(t_1+t_2+t_3)^2|(t_2-t_1)(t_3-t_2)(t_1-t_3)|$$

as required.

The normals will be concurrent iff the area of their triangle is 0. This is certainly true if $t_1+t_2+t_3=0$. In fact the only if is also true, since no 3 tangents can be concurrent.

Question (1992 STEP III Q9)

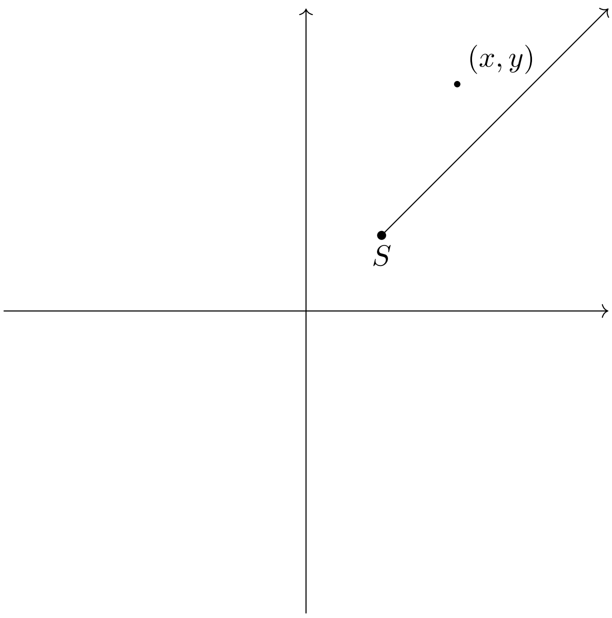
The straight line OSA, where O is the origin, bisects the angle between the positive x and y axes. The ellipse E has S as focus. In polar coordinates with S as pole and SA as the initial line, E has equation $\ell = r(1+e\cos\theta)$. Show that, at the point on E given by $\theta=\alpha$, the gradient of the tangent to the ellipse is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\alpha - \cos\alpha - e}{\sin\alpha + \cos\alpha + e}.$$

The points on E given by $\theta=\alpha$ and $\theta=\beta$ are the ends of a diameter of E. Show that

$$\tan(\alpha/2)\tan(\beta/2) = -\frac{1+e}{1-e}.$$

[**Hint.** A diameter of an ellipse is a chord through its centre.]



$$\ell = r(1+e\cos\theta)$$

$$\Rightarrow \quad 0 = \frac{\mathrm{d}r}{\mathrm{d}\theta}(1+e\cos\theta) - re\sin\theta$$

$$\Rightarrow \qquad \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{re\sin\theta}{1+e\cos\theta}$$

Suppose we consider the (x', y') plane, which is essentially the $x-y$ plan rotated by $45°$, then we would have

$$\begin{aligned}
\frac{\mathrm{d}y'}{\mathrm{d}x'} &= \frac{\frac{\mathrm{d}y'}{\mathrm{d}\theta}}{\frac{\mathrm{d}x'}{\mathrm{d}\theta}} \\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \\
&= \frac{\frac{re\sin\theta}{1+e\cos\theta}\sin\theta + r\cos\theta}{\frac{re\sin\theta}{1+e\cos\theta}\cos\theta - r\sin\theta} \\
&= \frac{re\sin^2\theta + r\cos\theta(1+e\cos\theta)}{re\sin\theta\cos\theta - r\sin\theta(1+e\cos\theta)} \\
&= \frac{\cos\theta + e\cos^2\theta + e\sin^2\theta}{-\sin\theta} \\
&= \frac{\cos\theta + e}{-\sin\theta}
\end{aligned}$$

Since our frame is rotated by $45°$ we need to consider the appropriate gradient for this. We know that $m = \tan\theta$ so $m' = \tan(\theta + 45°) = \frac{1+m}{1-m}$
therefore we should have

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 + \frac{\cos\theta+e}{-\sin\theta}}{1 - \frac{\cos\theta+e}{-\sin\theta}} \\
&= \frac{\cos\theta - \sin\theta + e}{-\sin\theta - \cos\theta - e} \\
&= \frac{\sin\theta - \cos\theta - e}{\sin\theta + \cos\theta + e}
\end{aligned}$$

As required.
The tangents at those points are parallel, therefore

$$\begin{aligned}
\frac{\cos\alpha + e}{\sin\alpha} &= \frac{\cos\beta + e}{\sin\beta} \\
\Rightarrow \qquad \frac{\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + e}{\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}} &= \frac{\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} + e}{\frac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}} \\
\frac{1-\tan^2\frac{\alpha}{2} + e(1+\tan^2\frac{\alpha}{2})}{2\tan\frac{\alpha}{2}} &= \frac{1-\tan^2\frac{\beta}{2} + e(1+\tan^2\frac{\beta}{2})}{2\tan\frac{\beta}{2}} \\
\frac{(1+e)+(e-1)\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}} &= \frac{(1+e)+(e-1)\tan^2\frac{\beta}{2}}{2\tan\frac{\beta}{2}} \\
\frac{(1+e)}{\tan\frac{\alpha}{2}} - (1-e)\tan\frac{\alpha}{2} &= \frac{(1+e)}{\tan\frac{\beta}{2}} - (1-e)\tan\frac{\beta}{2}
\end{aligned}$$

ie both $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are roots of a quadratic of the form $(1-e)x^2 - cx - (1+e)$ but this means the product of the roots is $-\frac{1+e}{1-e}$

Question (1994 STEP I Q5)

A parabola has the equation $y = x^2$. The points P and Q with coordinates (p, p^2) and (q, q^2) respectively move on the parabola in such a way that $\angle POQ$ is always a right angle.

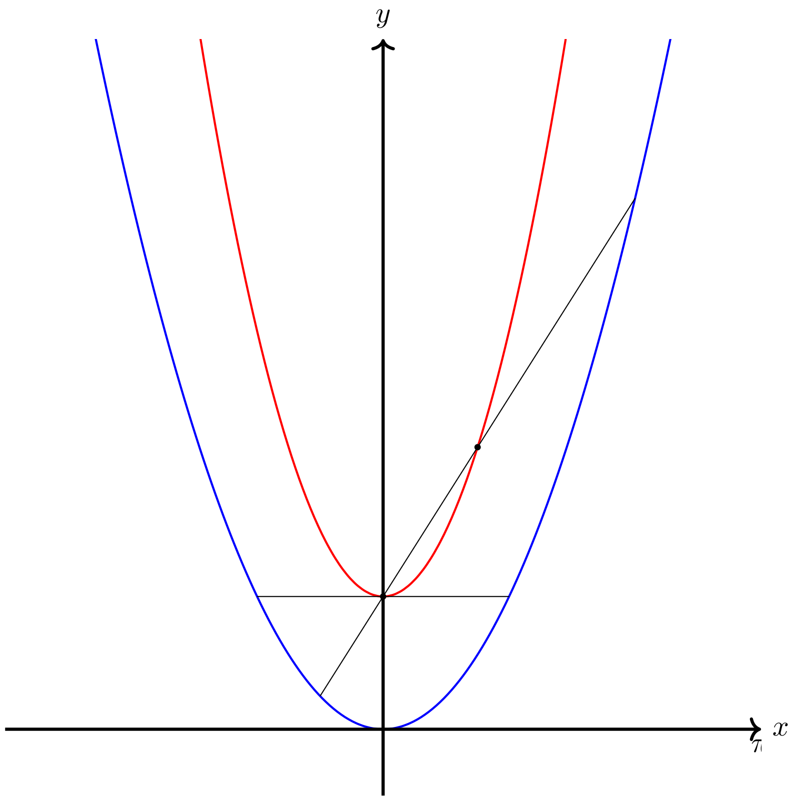
(i) Find and sketch the locus of the midpoint R of the chord PQ.

(ii) Find and sketch the locus of the point T where the tangents to the parabola at P and Q intersect.

(i) The line PO has gradient $\frac{p^2}{p} = p$ and teh line QO has gradient q, therefore we must have that $pq = -1$. Therefore, R is the point

$$\begin{aligned} R &= \left(\frac{p - \frac{1}{p}}{2}, \frac{p^2 + \frac{1}{p^2}}{2} \right) \\ &= \left(\frac{1}{2}\left(p - \frac{1}{p}\right), 2\left(\frac{1}{2}\left(p - \frac{1}{p}\right)\right)^2 + 1 \right) \\ &= \left(t, 2t^2 + 1\right) \end{aligned}$$

So we are looking at another parabola.

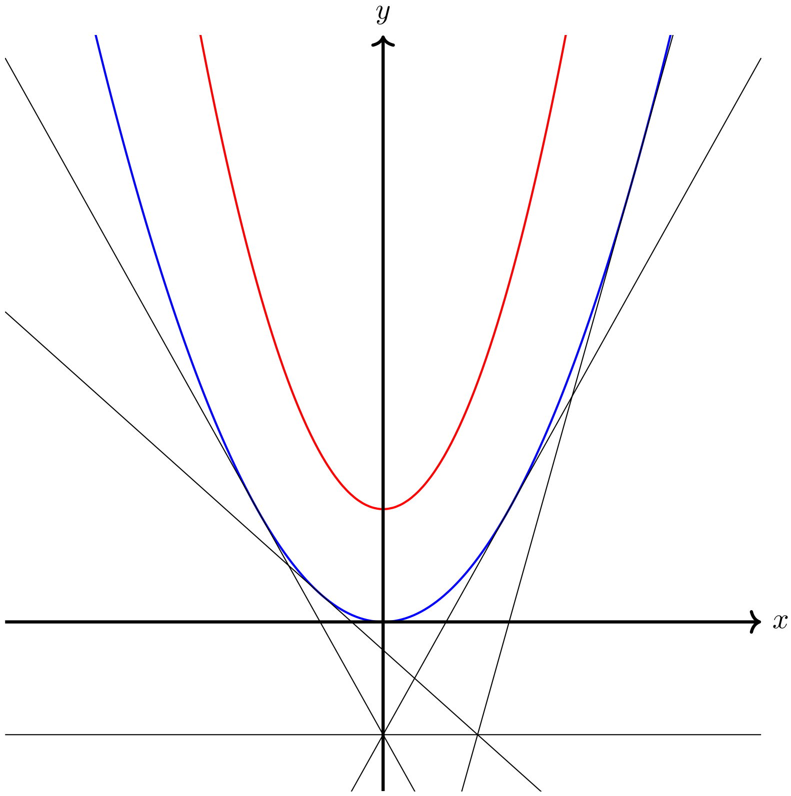


(ii) The tangents are $y = 2px + c$, ie $p^2 = 2p^2 + c$, ie $y = 2px - p^2$ so we have

$$y - 2px = -p^2$$

$$
\begin{aligned}
& & y - 2qx &= -q^2 \\
&\Rightarrow & (2p-2q)x &= p^2 - q^2 \\
&\Rightarrow & x &= \frac{1}{2}(p+q) \\
& & y &= p(p+q) - p^2 \\
& & y &= pq = -1
\end{aligned}
$$

Therefore $x = \frac{1}{2}(p - \frac{1}{p}), y = -1$, so we have the line $y = -1$ (the directrix)



Question (2003 STEP III Q7)

In the x–y plane, the point A has coordinates $(a\,,0)$ and the point B has coordinates $(0\,,b)$, where a and b are positive. The point P, which is distinct from A and B, has coordinates (s,t). X and Y are the feet of the perpendiculars from P to the x–axis and y–axis respectively, and N is the foot of the perpendicular from P to the line AB. Show that the coordinates $(x\,,y)$ of N are given by

$$x = \frac{ab^2 - a(bt - as)}{a^2 + b^2}\,, \quad y = \frac{a^2b + b(bt - as)}{a^2 + b^2}\,.$$

Show that, if $\left(\dfrac{t-b}{s}\right)\left(\dfrac{t}{s-a}\right) = -1$, then N lies on the line XY.

Give a geometrical interpretation of this result.

Question (2005 STEP III Q5)

Let P be the point on the curve $y = ax^2 + bx + c$ (where a is non-zero) at which the gradient is m. Show that the equation of the tangent at P is

$$y - mx = c - \frac{(m-b)^2}{4a}\,.$$

Show that the curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ (where a_1 and a_2 are non-zero) have a common tangent with gradient m if and only if

$$(a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + 4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2 = 0\,.$$

Show that, in the case $a_1 \neq a_2$, the two curves have exactly one common tangent if and only if they touch each other. In the case $a_1 = a_2$, find a necessary and sufficient condition for the two curves to have exactly one common tangent.

$$\begin{aligned} && y' &= 2ax + b \\ \Rightarrow && m &= 2ax_t + b \\ \Rightarrow && x_t &= \frac{m-b}{2a} \end{aligned}$$

Therefore we must have

$$\begin{aligned} mx_t &= 2ax_t^2 + bx_t \\ y - mx &= ax_t^2 + bx_t + c - mx_t \\ &= ax_t^2 + bx_t + c - (2ax_t^2 + bx_t) \\ &= c - ax_t^2 \\ &= c - a\left(\frac{m-b}{2a}\right)^2 \\ &= c - \frac{(m-b)^2}{4a} \end{aligned}$$

They will have a common tangent if and only if the constant terms are equal, ie

$$\begin{aligned} && c_1 - \frac{(m-b_1)^2}{4a_1} &= c_2 - \frac{(m-b_2)^2}{4a_2} \\ \Leftrightarrow && (c_1 - c_2) &= \frac{(m-b_1)^2}{4a_1} - \frac{(m-b_2)^2}{4a_2} \\ \Leftrightarrow && 4a_1a_2(c_1 - c_2) &= a_2(m-b_1)^2 - a_1(m-b_2)^2 \\ && &= (a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + a_2b_1^2 - a_1b_2^2 \end{aligned}$$

as required.

Treating this as a polynomial in m, we can see that the two curves will have exactly one common tangent iff $\Delta = 0$, ie:

$$\begin{aligned} 0 &= \Delta \\ &= (2(a_1b_2 - a_2b_1))^2 - 4(a_2 - a_1)(4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2) \end{aligned}$$

$$\begin{aligned}
&= 4a_1^2b_2^2 - 8a_1a_2b_1b_2 + 4a_2b_1^2 - 4a_2^2b_1^2 - 4a_1^2b_2^2 + 4a_1a_2(b_1^2+b_2^2) - 16(a_2-a_1)a_1a_2(c_2-c_1) \\
&= -8a_1a_2b_1b_2 + 4a_1a_2(b_1^2+b_2^2) - 16(a_2-a_1)a_1a_2(c_2-c_1) \\
&= a_1a_2(4(b_1-b_2)^2 - 16(a_2-a_1)(c_2-c_1)) \\
&= 4a_1a_2((b_2-b_1)^2 - 4(a_2-a_1)(c_2-c_1)
\end{aligned}$$

But this is just the discriminant of the difference, ie equivalent to the two parabolas just touching. (Assuming $a_1 - a_2 \neq 0$ and we do end up with a quadratic).

If $a_1 = a_2 = a$ then we need exactly one solution to $2a(b_1-b_2)m + 4a^2(c_2-c_1) + a(b_1^2 - b_2^2) = 0$, ie $b_1 \neq b_2$.

Question (2008 STEP III Q3)

The point $P(a\cos\theta\,,\, b\sin\theta)$, where $a > b > 0$, lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

The point $S(-ea\,,\,0)$, where $b^2 = a^2(1-e^2)$, is a focus of the ellipse. The point N is the foot of the perpendicular from the origin, O, to the tangent to the ellipse at P. The lines SP and ON intersect at T. Show that the y-coordinate of T is

$$\frac{b\sin\theta}{1+e\cos\theta}\,.$$

Show that T lies on the circle with centre S and radius a.

Find the gradient of the tangent of the ellipse at P:

$$\begin{aligned}
\frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2xb^2}{2ya^2} \\
&= -\frac{a\cos\theta b^2}{b\sin\theta a^2} \\
&= -\frac{b}{a}\cot\theta
\end{aligned}$$

Therefore the gradient of ON is $\frac{a}{b}\tan\theta$.

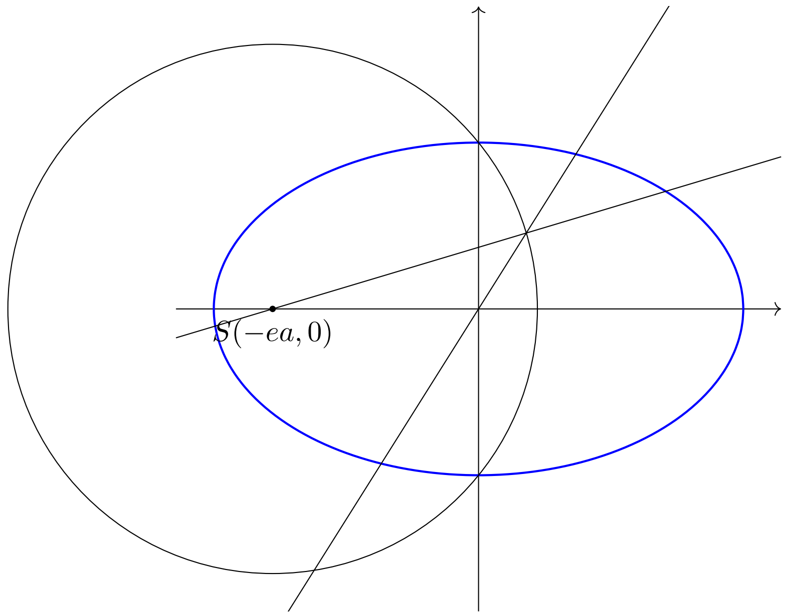
$$\begin{aligned}
y &= \frac{a}{b}\tan\theta x \\
\frac{y-0}{x-(-ea)} &= \frac{b\sin\theta - 0}{a\cos\theta - (-ea)} \\
y &= \frac{b\sin\theta}{a(e+\cos\theta)}(x+ea) \\
\Rightarrow \qquad y &= \frac{b\sin\theta}{a(\cos\theta+e)}\frac{b}{a}\cot\theta y + \frac{eb\sin\theta}{\cos\theta+e} \\
&= \frac{b^2\cos\theta}{a^2(\cos\theta+e)}y + \frac{eb\sin\theta}{\cos\theta+e} \\
\Rightarrow \qquad (\cos\theta+e)y &= (1-e^2)\cos\theta y + eb\sin\theta \\
e(1+e\cos\theta)y &= eb\sin\theta
\end{aligned}$$

$$\Rightarrow \qquad y = \frac{b\sin\theta}{1+e\cos\theta}$$
$$x = \frac{b\sin\theta}{1+e\cos\theta}\frac{b}{a}\cot\theta$$
$$= \frac{b^2\cos\theta}{a(1+e\cos\theta)}$$

Therefore $T\left(\frac{b^2\cos\theta}{a(1+e\cos\theta)}, \frac{b\sin\theta}{1+e\cos\theta}\right)$.
Finally, we can look at the distance of T from S

$$\begin{aligned}
d^2 &= \left(\frac{b^2\cos\theta}{a(1+e\cos\theta)} - (-ea)\right)^2 + \left(\frac{b\sin\theta}{1+e\cos\theta} - 0\right)^2 \\
&= \frac{\left(b^2\cos\theta + ea^2(1+e\cos\theta)\right)^2 + (ab\sin\theta)^2}{a^2(1+e\cos\theta)^2} \\
&= \frac{b^4\cos^2\theta + e^2a^4(1+e\cos\theta)^2 + 2ea^2b^2(1+e\cos\theta) + a^2b^2\sin^2\theta}{a^2(1+e\cos\theta)^2} \\
&= \frac{a^4(1-e^2)^2\cos^2\theta + e^2a^4(1+e\cos\theta)^2 + 2ea^2a^2(1-e^2)(1+e\cos\theta) + a^4(1-e^2)\sin^2\theta}{a^2(1+e\cos\theta)^2} \\
&= a^2\left(\frac{(1-e^2)^2\cos^2\theta + e^2(1+e\cos\theta)^2 + 2e(1-e^2)(1+e\cos\theta) + (1-e^2)(1-\cos^2\theta)}{(1+e\cos\theta)^2}\right) \\
&= a^2\left(\frac{e^2(1+e\cos\theta)^2 + (1-e^2)((1-e^2)\cos^2\theta + 2e(1+e\cos\theta) + (1-\cos^2\theta))}{(1+e\cos\theta)^2}\right) \\
&= a^2\left(\frac{e^2(1+e\cos\theta)^2 + (1-e^2)(1+e\cos\theta)^2}{(1+e\cos\theta)^2}\right) \\
&= a^2
\end{aligned}$$

Therefore a circle radius a centre S.



Question (2014 STEP III Q3) **(i)** The line L has equation $y = mx + c$, where $m > 0$ and $c > 0$. Show that, in the case $mc > a > 0$, the shortest distance between L and the parabola $y^2 = 4ax$ is

$$\frac{mc - a}{m\sqrt{m^2 + 1}}.$$

What is the shortest distance in the case that $mc \leq a$?

(ii) Find the shortest distance between the point $(p, 0)$, where $p > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the value of p/a. [*You may wish to use the parametric coordinates* $(at^2, 2at)$ *of points on the parabola.*] Hence find the shortest distance between the circle $(x - p)^2 + y^2 = b^2$, where $p > 0$ and $b > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the values of p, a and b.

Question (2016 STEP III Q2)

The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P.

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point that is independent of the choice of P.

(iii) Let T be the point of intersection of OP and QR, where O is the coordinate origin. Show that T lies on a line that is independent of the choice of P. Show further that the distance from the x-axis to T is less than $\dfrac{a}{\sqrt{2}}$.

(i)

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

Therefore we must have

$$\underbrace{-\frac{2aq}{2a}}_{\text{gradient of normal}} = \underbrace{\frac{2ap - 2aq}{ap^2 - aq^2}}_{\Delta y/\Delta x}$$

$$\Rightarrow \quad -q = \frac{2}{p + q}$$

$$0 = 2 + pq + q^2$$

(ii) We must have that q, r are the two roots of $x^2 + px + 2 = 0$

QR has the equation:

$$\begin{aligned}
\frac{y-2aq}{x-aq^2} &= \frac{2ar-2aq}{ar^2-aq^2} \\
\Rightarrow \quad \frac{y-2aq}{x-aq^2} &= \frac{2}{r+q} \\
\Rightarrow \quad y &= \frac{2}{q+r}(x-aq^2) + 2aq \\
y &= -\frac{2}{p}x + 2a\left(q - \frac{q^2}{q+r}\right) \\
y &= -\frac{2}{p}x + 2a\frac{qr}{q+r} \\
y &= -\frac{2}{p}x - 2a\frac{2}{p} \\
y &= -\frac{2}{p}(x+2a)
\end{aligned}$$

Therefore the point $(-2a, 0)$ lies on all such lines.

(iii) OP has equation $y = \frac{2}{p}x$

$$\begin{aligned}
y &= \frac{2}{p}x \\
y &= -\frac{2}{p}(x+2a) \\
2y &= -\frac{4a}{p} \\
\Rightarrow \quad y &= -\frac{2a}{p} \\
x &= -a
\end{aligned}$$

Therefore $T\left(-a, -\frac{2a}{p}\right)$ always lies on the line $x = -a$

The distance to the x-axis from T is $\frac{2a}{|p|}$. We need to show that p can't be too small. Specifically $x^2 + px + 2 = 0$ must have 2 real roots, ie $\Delta = p^2 - 8 \geq 0 \Rightarrow |p| \geq 2\sqrt{2}$, ie $\frac{2a}{|p|} \leq \frac{2a}{2\sqrt{2}} = \frac{a}{\sqrt{2}}$ as required.

Question (1989 STEP II Q2)

Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{for small } x,$$
$$x\cot x = 1 + \sum_{n=1}^{\infty} b_n x^n \quad \text{for small } x \text{ and not zero.}$$

Using the relation

$$\cot x - \tan x = 2\cot 2x, \qquad (*)$$

or otherwise, prove that $a_{n-1} = (1-2^n)b_n$, for $n \geqslant 1$. Let

$$x\text{cosec}x = 1 + \sum_{n=1}^{\infty} c_n x^n \quad \text{for small } x \neq 0.$$

Using a relation similar to $(*)$ involving $2\text{cosec}2x$, or otherwise, prove that

$$c_n = \frac{2^{n-1}-1}{2^n-1}\frac{1}{2^{n-1}}a_{n-1} \qquad (n \geqslant 1).$$

$$\begin{aligned}
&& \cot x - \tan x &= 2\cot 2x \\
\Rightarrow && x\cot x - x\tan x &= 2x\cot 2x \\
\Rightarrow && 1 + \sum_{n=1}^{\infty} b_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} &= 1 + \sum_{n=1}^{\infty} b_n (2x)^n \\
\Rightarrow && \sum_{n=1}^{\infty} (1-2^n) b_n x^n &= \sum_{n=1}^{\infty} a_{n-1} x^n \\
\Rightarrow && a_{n-1} &= (1-2^n)b_n \quad \text{if } n \geq 1
\end{aligned}$$

$$\cot x + \tan x = 22x$$

So

$$\begin{aligned}
&& \cot x + \tan x &= 22x \\
\Rightarrow && x\cot x + x\tan x &= 2x2x \\
\Rightarrow && 1 + \sum_{n=1}^{\infty} b_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} &= 1 + \sum_{n=1}^{\infty} c_n (2x)^n \\
\Rightarrow && \sum_{n=1}^{\infty} \frac{1}{1-2^n} a_{n-1} + \sum_{n=1}^{\infty} a_{n-1} x^n &= \sum_{n=1}^{\infty} 2^n c_n x^n \\
\Rightarrow && c_n &= \frac{1}{2^n}\left(1 + \frac{1}{1-2^n}\right) a_{n-1} \\
&& &= \frac{1}{2^n}\frac{2^n-2}{2^n-1} a_{n-1} \\
&& &= \frac{1}{2^{n-1}}\frac{2^{n-1}-1}{2^n-1} a_{n-1}
\end{aligned}$$

Question (1990 STEP III Q7)

The points $P\,(0, a)$, $Q\,(a, 0)$ and $R\,(a, -a)$ lie on the curve C with cartesian equation

$$xy^2 + x^3 + a^2y - a^3 = 0, \qquad \text{where } a > 0.$$

At each of P, Q and R, express y as a Taylor series in h, where h is a small increment in x, as far as the term in h^2. Hence, or otherwise, sketch the shape of C near each of these points. Show that, if (x, y) lies on C, then

$$4x^4 - 4a^3x - a^4 \leqslant 0.$$

Sketch the graph of $y = 4x^4 - 4a^3x - a^4$. Given that the y-axis is an asymptote to C, sketch the curve C.

$$\begin{aligned}
& & 0 &= xy^2 + x^3 + a^2y - a^3 \\
\frac{\mathrm{d}}{\mathrm{d}x}: & & 0 &= y^2 + 2xyy' + 3x^2 + a^2y' \\
\Rightarrow & & y' &= -\frac{y^2 + 3x^2}{a^2 + 2xy}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}: & & 0 &= 2yy' + 2yy' + 2x(y')^2 + 2xyy'' + 6x + a^2y'' \\
\Rightarrow & & y'' &= -\frac{4yy' + 2x(y')^2 + 6x}{a^2 + 2xy}
\end{aligned}$$

$$\begin{aligned}
P: & & y &= a \\
& & y' &= -\frac{a^2}{a^2} = -1 \\
& & y'' &= -\frac{-4a}{a^2} = \frac{4}{a} \\
\Rightarrow & & y &\approx a - h + \frac{2}{a}h^2
\end{aligned}$$

$$\begin{aligned}
Q: & & y &= 0 \\
& & y' &= -\frac{3a^2}{a^2} = -3 \\
& & y'' &= -\frac{18a + 6a}{a^2} = -\frac{24}{a} \\
\Rightarrow & & y &\approx 0 - 3h - \frac{12}{a}h
\end{aligned}$$

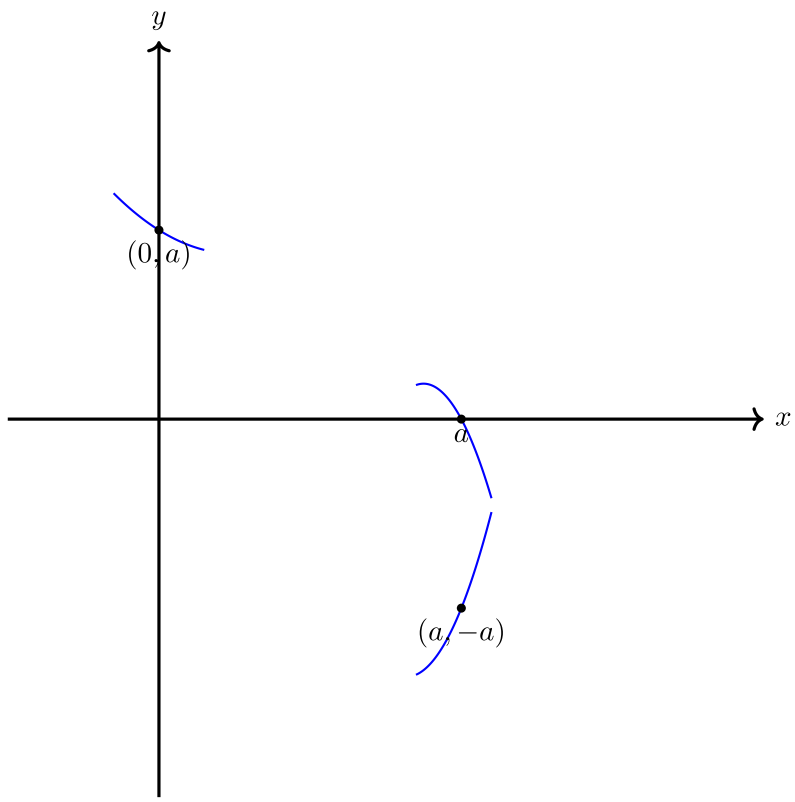
$$\begin{aligned}
R: & & y &= -a \\
& & y' &= -\frac{a^2 + 3a^2}{a^2 - 2a^2} = 4 \\
& & y'' &= -\frac{-16a + 32a + 6a}{a^2 - 2a^2} = \frac{22}{a} \\
\Rightarrow & & y &\approx -a + 4h + \frac{11}{a}h^2
\end{aligned}$$

Alternatively:

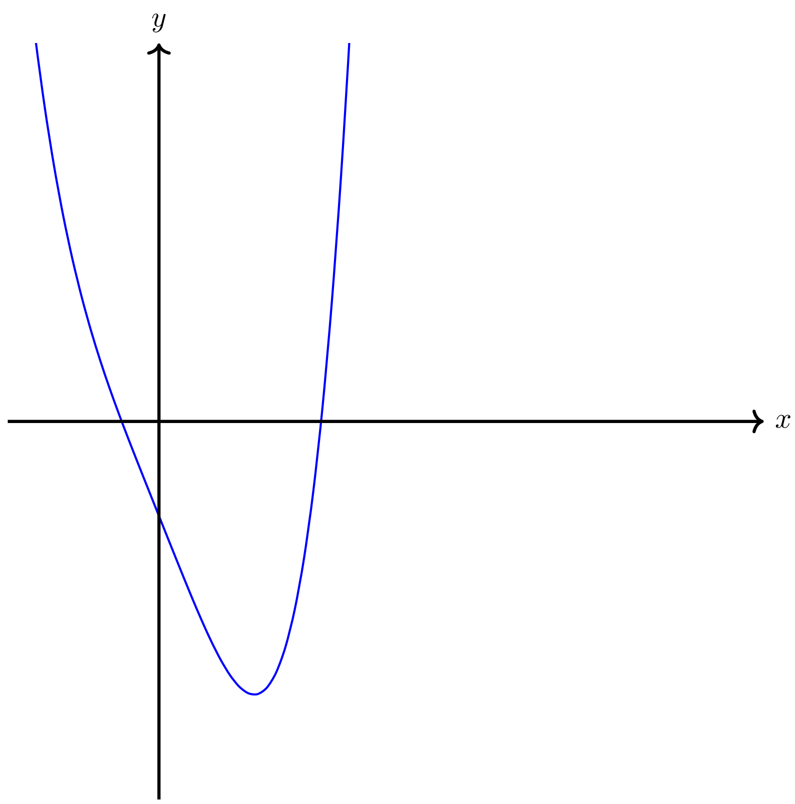
$$
\begin{aligned}
& 0 = xy^2 + x^3 + a^2y - a^3 \\
P(0,a): \quad & y \approx a + c_1h + c_2h^2 \\
& 0 = h(a + c_1h)^2 + a^2(a + c_1h + c_2h^2) - a^3 \\
& \quad = a^3 - a^3 + (a^2 + a^2c_1)h + (2ac_1 + a^2c_2)h^2 \\
\Rightarrow \quad & c_1 = -1, c_2 = \frac{2}{a} \\
\Rightarrow \quad & y \approx a - h + \frac{2}{a}h^2
\end{aligned}
$$

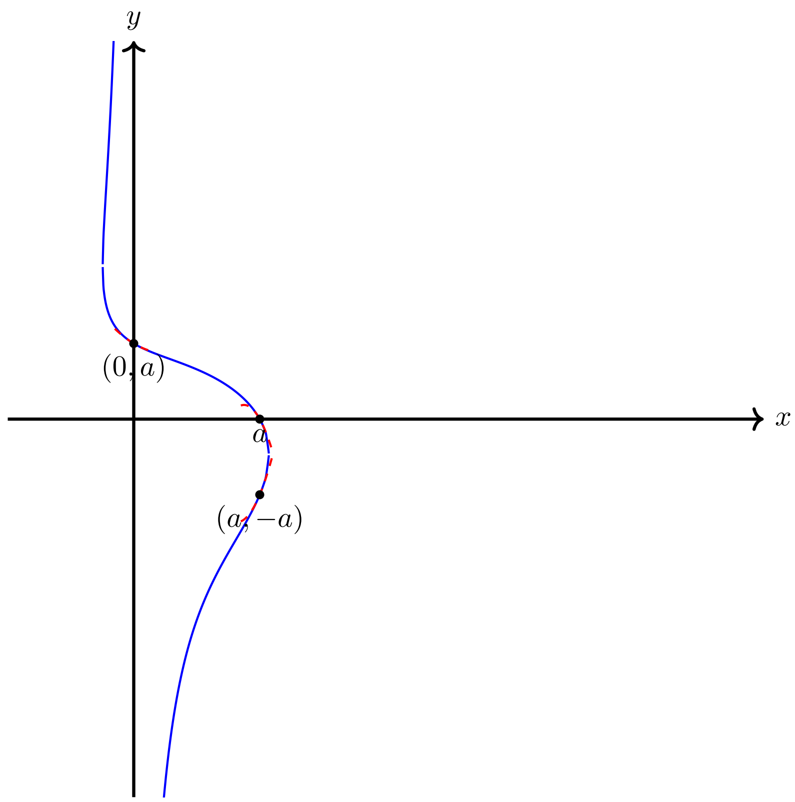
$$
\begin{aligned}
Q(a,0): \quad & y \approx c_1h + c_2h^2 \\
& 0 = (a + h)(c_1h)^2 + (a + h)^3 + a^2(c_1h + c_2h^2) - a^3 \\
& \quad = a^3 - a^3 + (3a^2 + a^2c_1)h + (ac_1^2 + 3a + a^2c_2)h^2 + \cdots \\
\Rightarrow \quad & c_1 = -3, c_2 = -\frac{12}{a} \\
\Rightarrow \quad & y \approx -3h - \frac{12}{a}h
\end{aligned}
$$

$$
\begin{aligned}
R(a,-a): \quad & y \approx -a + c_1h + c_2h^2 \\
& 0 = (a + h)(-a + c_1h + c_2h^2)^2 + (a + h)^3 + a^2(-a + c_1h + c_2h^2) - a^3 \\
& \quad = (a^2 - 2a^2c_1 + 3a^2 + a^2c_1)h + (-2ac_1 + c_1^2 + \cdots)h^2 \\
\Rightarrow \quad & c_1 = 4, c_2 = \frac{11}{a} \\
\Rightarrow \quad & y \approx -a + 4h + \frac{11}{a}
\end{aligned}
$$



If (x, y) lies on the curve, then viewing it as a quadratic in y we must have $\Delta = (a^2)^2 - 4 \cdot x \cdot (x^3 - a^3) \geq 0 \Rightarrow a^4 - 4x^4 + 4xa^3 \geq 0 \Rightarrow 4x^4 - 4a^3x - a^4 \leq 0$





Question (1991 STEP III Q1) **(i)** Evaluate

$$\sum_{r=1}^{n} \frac{6}{r(r+1)(r+3)}.$$

(ii) Expand $\ln(1+x+x^2+x^3)$ as a series in powers of x, where $|x|<1$, giving the first five non-zero terms and the general term.

(iii) Expand $e^{x\ln(1+x)}$ as a series in powers of x, where $-1<x\leqslant 1$, as far as the term in x^4.

(i)

$$\begin{aligned} \frac{6}{r(r+1)(r+3)} &= \frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+3} \\ \Rightarrow \qquad \sum_{r=1}^{n} \frac{6}{r(r+1)(r+3)} &= \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+3} \right) \\ &= \sum_{r=1}^{n} \frac{2}{r} - \sum_{r=1}^{n} \frac{3}{r+1} + \sum_{r=1}^{n} \frac{1}{r+3} \\ &= \sum_{r=1}^{n} \frac{2}{r} - \sum_{r=2}^{n+1} \frac{3}{r} + \sum_{r=3}^{n+2} \frac{1}{r} \\ &= \frac{2}{1} + \frac{2}{2} - \frac{3}{2} - \frac{3}{n+1} + \frac{1}{n+1} + \frac{1}{n+2} \end{aligned}$$

$$= \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$$

(ii)

$$\begin{aligned}
\ln(1+x+x^2+x^3) &= \ln\left(\frac{1-x^4}{1-x}\right) \\
&= \ln(1-x^4) - \ln(1-x) \\
&= \sum_{k=1}^{\infty} -\frac{x^{4k}}{k} - \sum_{k=1}^{\infty} -\frac{x^k}{k} \\
&= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{5}x^5 + \cdots \\
&= \sum_{k=1}^{\infty} a_k x^k
\end{aligned}$$

Where $a_k = \frac{1}{k}$ if $k \neq 0 \pmod 4$ otherwise $a_k = -\frac{3}{k}$ if $k \equiv 0 \pmod 4$

(iii)

$$\begin{aligned}
\exp(x\ln(1+x)) &= \exp\left(x\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right)\right) \\
&= \exp\left(x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4\right) \\
&= 1 + \left(x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4\right) + \frac{1}{2}\left(x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4\right)^2 + \cdots \\
&= 1 + x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{1}{2}x^4 + \cdots \\
&= 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 + \cdots
\end{aligned}$$

Question (1994 STEP III Q5)

The function f is given by $f(x) = \sin^{-1} x$ for $-1 < x < 1$. Prove that

$$(1-x^2)f''(x) - xf'(x) = 0.$$

Prove also that

$$(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) - n^2 f^{(n)}(x) = 0,$$

for all $n > 0$, where $f^{(n)}$ denotes the nth derivative of f. Hence express $f(x)$ as a Maclaurin series. The function g is given by

$$g(x) = \ln\sqrt{\frac{1+x}{1-x}},$$

for $-1 < x < 1$. Write down a power series expression for $g(x)$, and show that the coefficient of x^{2n+1} is greater than that in the expansion of f, for each $n > 0$.

None

Question (1997 STEP III Q1) **(i)** By considering the series expansion of $(x^2 + 5x + 4)\, e^x$ show that

$$10\, e = 4 + \frac{3^2}{1!} + \frac{4^2}{2!} + \frac{5^2}{3!} + \cdots .$$

(ii) Show that

$$5\, e = 1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots .$$

(iii) Evaluate

$$1 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \cdots .$$

(i)

$$\begin{aligned}
(x^2 + 5x + 4)e^x &= \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+2} + \sum_{k=0}^{\infty} \frac{5}{k!} x^{k+1} + \sum_{k=0}^{\infty} \frac{4}{k!} x^k \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{k!} + \frac{5}{(k+1)!} + \frac{4}{(k+2)!} \right) x^{k+2} + 5x + 4 + 4x \\
&= 4 + 9x + \sum_{k=0}^{\infty} \left(\frac{(k+2)(k+1)}{(k+2)!} + \frac{5(k+2)}{(k+2)!} + \frac{4}{(k+2)!} \right) x^{k+2} \\
&= 4 + 9x + \sum_{k=0}^{\infty} \left(\frac{k^2 + 3k + 2 + 5k + 10 + 4}{(k+2)!} \right) x^{k+2} \\
&= 4 + 9x + \sum_{k=0}^{\infty} \frac{(k+4)^2}{(k+2)!} x^{k+2} \\
&= 4 + 9x + \sum_{k=2}^{\infty} \frac{(k+2)^2}{k!} x^k
\end{aligned}$$

So when $x = 1$ we have

$$10e = 4 + \frac{3^2}{1!} + \frac{4^2}{2!} + \frac{5^2}{3!} + \cdots$$

(ii)

$$\begin{aligned}
(x^2 + 3x + 1)e^x &= \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+2} + \sum_{k=0}^{\infty} 3\frac{1}{k!} x^{k+1} + \sum_{k=0}^{\infty} \frac{1}{k!} x^k \\
&= 1 + 3x + \sum_{k=1}^{\infty} \left(\frac{1}{(k-1)!} + \frac{3}{k!} + \frac{1}{(k+1)!} \right) x^{k+1} \\
&= 1 + 3x + \sum_{k=1}^{\infty} \frac{(k+1)k + 3(k+1) + 1}{(k+1)!} x^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= 1 + 3x + \sum_{k=1}^{\infty} \frac{k^2 + 4k + 4}{(k+1)!} x^{k+1} \\
&= 1 + 3x + \sum_{k=0}^{\infty} \frac{(k+2)^2}{(k+1)!} x^{k+1} \\
&= 1 + 3x + \sum_{k=1}^{\infty} \frac{(k+1)^2}{k!} x^{k}
\end{aligned}
$$

Plugging in $x = 1$ we get the desired result.

(iii)

$$
\begin{aligned}
& & xe^x &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} \\
x\frac{\mathrm{d}}{\mathrm{d}x}: & & x(1+x)e^x &= \sum_{k=0}^{\infty} \frac{(k+1)x^{k+1}}{k!} \\
x\frac{\mathrm{d}}{\mathrm{d}x}: & & x(x(1+x)+1+2x)e^x &= \sum_{k=0}^{\infty} \frac{(k+1)^2x^{k+1}}{k!} \\
& & (x^3+3x^2+x)e^x &= \sum_{k=0}^{\infty} \frac{(k+1)^2x^{k+1}}{k!} \\
\frac{\mathrm{d}}{\mathrm{d}x}: & & e^x(x^3+3x^2+x+3x^2+6x+1) &= \sum_{k=0}^{\infty} \frac{(k+1)^3x^{k}}{k!} \\
\Rightarrow & & 15e &= 1 + \frac{2^3}{1!} + \frac{3^3}{2!} + \cdots
\end{aligned}
$$

Question (1998 STEP III Q5)

The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r ,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix. Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single 2×2 matrix, where θ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$. Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Express similarly $\exp(s\mathbf{N})$, where s is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$. For which values of θ does

$$\exp(s\mathbf{N})\ \exp(\theta\mathbf{M}) \ = \ \exp(\theta\mathbf{M})\ \exp(s\mathbf{N})$$

for all s? Interpret this fact geometrically.

$$\begin{aligned}
\mathbf{M}^2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0\cdot 0 + (-1)\cdot 1 & 0\cdot(-1) + (-1)\cdot 0 \\ 1\cdot 0 + 0\cdot 1 & 1\cdot(-1) + 0\cdot 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= -\mathbf{I}
\end{aligned}$$

$$\begin{aligned}
\exp(\theta\mathbf{M}) &= \sum_{r=0}^{\infty} \frac{1}{r!}(\theta\mathbf{M})^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!}\theta^r\mathbf{M}^r \\
&= \cos\theta\mathbf{I} + \sin\theta\mathbf{M} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\end{aligned}$$

This is a rotation of θ degrees about the origin.

$$\begin{aligned}
\mathbf{N}^2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \exp(s\mathbf{N}) &= \sum_{r=0}^{\infty} \frac{1}{r!}(s\mathbf{N})^r \\
&= \mathbf{I} + s\mathbf{N} \\
&= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

This is a shear, leaving the y-axis invariant, sending $(1,1)$ to $(1+s,1)$.
Suppose those matrices commute, for all s, ie

$$\begin{aligned}
\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \\
\Rightarrow \qquad \begin{pmatrix} \cos\theta - s\sin\theta & -\sin\theta + s\cos\theta \\ \sin\theta & \cos\theta \end{pmatrix} &= \begin{pmatrix} \cos\theta & s\cos\theta - \sin\theta \\ \sin\theta & s\sin\theta + \cos\theta \end{pmatrix} \\
\Rightarrow \qquad \sin\theta &= 0 \\
\Rightarrow \qquad \theta &= n\pi, n \in \mathbb{Z}
\end{aligned}$$

Clearly it doesn't matter when we do nothing. If we are rotating by π then it also doesn't matter which order we do it in as the stretch happens in both directions equally.

Question (1998 STEP III Q7)

Sketch the graph of $\mathrm{f}(s) = \mathrm{e}^s(s-3)+3$ for $0 \leq s < \infty$. Taking $\mathrm{e} \approx 2.7$, find the smallest positive integer, m, such that $\mathrm{f}(m) > 0$. Now let

$$\mathrm{b}(x) = \frac{x^3}{\mathrm{e}^{x/T}-1}$$

where T is a positive constant. Show that $\mathrm{b}(x)$ has a single turning point in $0 < x < \infty$. By considering the behaviour for small x and for large x, sketch $\mathrm{b}(x)$ for $0 \leq x < \infty$. Let

$$\int_0^\infty \mathrm{b}(x)\,\mathrm{d}x = B,$$

which may be assumed to be finite. Show that $B = KT^n$ where K is a constant, and n is an integer which you should determine. Given that $B \approx 2\displaystyle\int_0^{Tm} \mathrm{b}(x)\,\mathrm{d}x$, use your graph of $\mathrm{b}(x)$ to find a rough estimate for K.

None

Question (2001 STEP III Q1)

Given that $y = \ln(x + \sqrt{x^2+1})$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+1}}$. Prove by induction that, for $n \geq 0$,

$$\left(x^2+1\right) y^{(n+2)} + (2n+1)\, xy^{(n+1)} + n^2 y^{(n)} = 0 \, ,$$

where $y^{(n)} = \dfrac{d^n y}{dx^n}$ and $y^{(0)} = y$. Using this result in the case $x = 0$, or otherwise, show that the Maclaurin series for y begins

$$x - \frac{x^3}{6} + \frac{3x^5}{40}$$

and find the next non-zero term.

$$\begin{aligned} y &= \ln(x + \sqrt{x^2+1}) \\ \Rightarrow \qquad \frac{dy}{dx} &= \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x + \sqrt{x^2+1}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

Note that $y^{(2)} = -\dfrac{1}{2}\dfrac{2x}{(x^2+1)^{3/2}} = -\dfrac{x}{(x^2+1)^{3/2}}$, and in particular $(x^2+1)y^{(2)} + xy^{(1)} = 0$.

Now applying Leibnitz formula:

$$\begin{aligned} 0 &= \left((x^2+1)y^{(2)} + xy^{(1)}\right)^{(n)} \\ &= \left((x^2+1)y^{(2)}\right)^{(n)} + \left(xy^{(1)}\right)^{(n)} \\ &= (x^2+1)y^{(n+2)} + n2xy^{(n+1)} + \binom{n}{2}2y^{(n)} + xy^{(n+1)} + ny^{(n)} \\ &= (x^2+1)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2-n+n)y^{(n)} \\ &= (x^2+1)y^{(n+2)} + (2n+1)xy^{(n+1)} + n^2y^{(n)} \end{aligned}$$

as required.

When $x = 0$:

$$\begin{aligned} y(0) &= \ln(0 + \sqrt{0^2+1}) \\ &= \ln 1 = 0 \\ y'(0) &= \frac{1}{\sqrt{0^2+1}} = 1 \\ y^{(n+2)} &= -n^2 y^{(n)} \end{aligned}$$

$$\begin{aligned} y^{(2k)} &= 0 \\ y^{(3)} &= -1 \\ y^{(5)} &= 3^2 \\ y^{(7)} &= -5^2 \cdot 3^2 \end{aligned}$$

Therefore the Maclaurin series about $x = 0$ is

$$\begin{aligned} y &= x - \frac{1}{3!}x^3 + \frac{3^2}{5!}x^5 - \frac{3^2 \cdot 5^2}{7!}x^7 + \cdots \\ &= x - \frac{1}{6}x^3 + \frac{3}{1 \cdot 2 \cdot 4 \cdot 5}x^5 - \frac{5}{1 \cdot 2 \cdot 4 \cdot 2 \cdot 7}x^7 + \cdots \\ &= x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{56}x^7 + \cdots \end{aligned}$$

Question (2006 STEP III Q3) **(i)** Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \text{ and } \cot x = \frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n$$

for $0 < x < \frac{1}{2}\pi$. Explain why $a_n = 0$ for even n. Prove the identity

$$\cot x - \tan x \equiv 2 \cot 2x$$

and show that

$$a_n = (1 - 2^{n+1})b_n \,.$$

(ii) Let $cosec\, x = \frac{1}{x} + \sum_{n=0}^{\infty} c_n x^n$ for $0 < x < \frac{1}{2}\pi$. By considering $\cot x + \tan x$, or otherwise, show that

$$c_n = (2^{-n} - 1)b_n \,.$$

(iii) Show that

$$\left(1 + x\sum_{n=0}^{\infty} b_n x^n\right)^2 + x^2 = \left(1 + x\sum_{n=0}^{\infty} c_n x^n\right)^2 .$$

Deduce from this and the previous results that $a_1 = 1$, and find a_3.

(i) Since $\tan(-x) = -\tan x$, tan is an odd function, and in particular all it's even coefficients are zero.

$$\begin{aligned} 2\cot 2x &\equiv \frac{2cos2x}{\sin 2x} \\ &\equiv \frac{2(\cos^2 x - \sin^2 x)}{2 \sin x \cos x} \\ &\equiv \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} \end{aligned}$$

$$\equiv \cot x - \tan x$$

Therefore

$$\begin{aligned}
\underbrace{\frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n}_{\cot x} - \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\tan x} &= 2\left(\underbrace{\frac{1}{2x} + \sum_{n=0}^{\infty} b_n (2x)^n}_{\cot 2x}\right) \\
\Rightarrow \qquad \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} b_n x^n - 2\sum_{n=0}^{\infty} b_n (2x)^n \\
&= \sum_{n=0}^{\infty} b_n (1 - 2^{n+1}) x^n \\
[x^n] : \qquad a_n &= (1 - 2^{n+1}) b_n
\end{aligned}$$

(ii)

$$\begin{aligned}
\cot x + \tan x &= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\
&= \frac{1}{\sin x \cos x} \\
&= 22x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \underbrace{\frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n}_{\cot x} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\tan x} &= 2\left(\underbrace{\frac{1}{2x} + \sum_{n=0}^{\infty} c_n (2x)^n}_{2x}\right) \\
\Rightarrow \qquad \sum_{n=0}^{\infty} 2^{n+1} c_n x^n &= \sum_{n=0}^{\infty} (a_n + b_n) x^n \\
&= \sum_{n=0}^{\infty} \left((1 - 2^{n+1}) b_n + b_n\right) x^n \\
&= \sum_{n=0}^{\infty} \left(2 - 2^{n+1}\right) b_n x^n \\
[x^n] : \qquad c_n &= (2^{-n} - 1) b_n
\end{aligned}$$

(iii)

$$\begin{aligned}
\cot^2 x + 1 &=^2 x \\
\Rightarrow \qquad x^2 \cot^2 x + x^2 &= x^{22} x \\
\Rightarrow \qquad x^2 \left(\underbrace{\frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n}_{\cot x}\right)^2 + x^2 &= x^2 \left(\underbrace{\frac{1}{x} + \sum_{n=0}^{\infty} c_n x^n}_{x}\right)^2
\end{aligned}$$

$$\Rightarrow \qquad \left(1 + x\sum_{n=0}^{\infty} b_n x^n\right)^2 + x^2 = \left(1 + x\sum_{n=0}^{\infty} c_n x^n\right)^2$$

$$\begin{aligned}
\Rightarrow \qquad & \big(1 + x(b_1 x + b_3 x^3 + \cdots)\big)^2 + x^2 = \big(1 + x(c_1 x + c_3 x^3 + \cdots)\big)^2 \\
\Rightarrow \qquad & 1 + (1 + 2b_1)x^2 + (2b_3 + b_1^2)x^4 + \cdots = 1 + 2c_1 x^2 + (2c_3 + c_1^2)x^4 + \cdots \\
\Rightarrow \qquad & 1 + 2b_1 = 2(2^{-1} - 1)b_1 \\
\Rightarrow \qquad & b_1 = -\frac{1}{3} \\
\Rightarrow \qquad & a_1 = (1 - 2^2)(-\tfrac{1}{3}) = 1 \\
& c_1 = \frac{1}{6} \\
\Rightarrow \qquad & 2b_3 + \frac{1}{9} = 2c_3 + \frac{1}{36} \\
\Rightarrow \qquad & 2b_3 - 2(2^{-3} - 1)b_3 = -\frac{1}{12} \\
\Rightarrow \qquad & \frac{15}{4} b_3 = -\frac{1}{12} \\
\Rightarrow \qquad & b_3 = -\frac{1}{45} \\
\Rightarrow \qquad & a_3 = -(1 - 2^4)\frac{1}{45} = \frac{1}{3}
\end{aligned}$$

Question (2006 STEP III Q4)

The function f satisfies the identity

$$f(x) + f(y) \equiv f(x + y) \qquad (*)$$

for all x and y. Show that $2f(x) \equiv f(2x)$ and deduce that $f''(0) = 0$. By considering the Maclaurin series for $f(x)$, find the most general function that satisfies $(*)$. [*Do not consider issues of existence or convergence of Maclaurin series in this question.*]

(i) By considering the function , defined by $\ln\big(g(x)\big) = (x)$, find the most general function that, for all x and y, satisfies the identity

$$g(x)g(y) \equiv g(x + y)\,.$$

(ii) By considering the function H, defined by $h(\mathrm{e}^u) = H(u)$, find the most general function that satisfies, for all positive x and y, the identity

$$h(x) + h(y) \equiv h(xy)\,.$$

(iii) Find the most general function t that, for all x and y, satisfies the identity

$$t(x) + t(y) \equiv t(z)\,,$$

where $z = \dfrac{x + y}{1 - xy}$.

$$\begin{aligned} 2f(x) &\equiv f(x) + f(x) \\ &\equiv f(x+x) \\ &\equiv f(2x) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad 2f(0) &= f(0) \\ \Rightarrow \qquad f(0) &= 0 \\ f''(0) &= \lim_{h\to 0} \frac{f(2h) - 2f(0) + f(-2h)}{h^2} \\ &= \lim_{h\to 0} \frac{f(2h) + f(-2h)}{h^2} \\ &= \lim_{h\to 0} \frac{f(0)}{h^2} \\ &= 0 \\ \Rightarrow \qquad f''(0) &= 0 \end{aligned}$$

If $f(x)$ satisfies the equation, then $f'(x)$ satisfies the equation. In particular this means that $f^{(n)}(0) = 0$ for all $n \geq 2$. Therefore the only non-zero term in the Maclaurin series is x^1. Therefore $f(x) = cx$

(i) Suppose $g(x)g(y) \equiv g(x+y)$, then if $G(x) = \ln g(x)$ we must have $G(x) + G(y) \equiv G(x+y)$, ie $G(x) = cx \Rightarrow g(x) = e^{cx}$

(ii) Suppose $h(x)+h(y) \equiv h(xy)$, then if $h(e^u) = H(u)$ we must have that $H(u)+H(v) \equiv h(e^u) + h(e^v) \equiv h(e^{u+v}) \equiv H(u+v)$.Therefore $H(u) = cu$, ie $h(e^u) = cu$ or $h(x) = h(e^{\ln x}) = c\ln x$.

(iii) Finally if $t(x) + t(y) \equiv t(z)$, the considering $T(w) = t(\tan w)$ then $T(x) + T(y) \equiv t(\tan x) + t(\tan y) \equiv t(\frac{\tan x + \tan y}{1 - \tan x \tan y}) \equiv t(\tan(x+y)) \equiv T(x+y)$. Therefore $T(x) = cx$ Therefore $t(\tan w) = cw \Rightarrow t(x) = c\tan^{-1} x$

$e^t = 1 + t + \frac{t^2}{2} + \cdots$, therefore

$$\begin{aligned} \lim_{t\to 0} f(t) &= \lim_{t\to 0} \frac{t}{e^t - 1} \\ &= \lim_{t\to 0} \frac{t}{t + \frac{t^2}{2} + o(t^3)} \\ &= \lim_{t\to 0} \frac{1}{1 + \frac{t}{2} + o(t^2)} \\ &\to 1 \end{aligned}$$

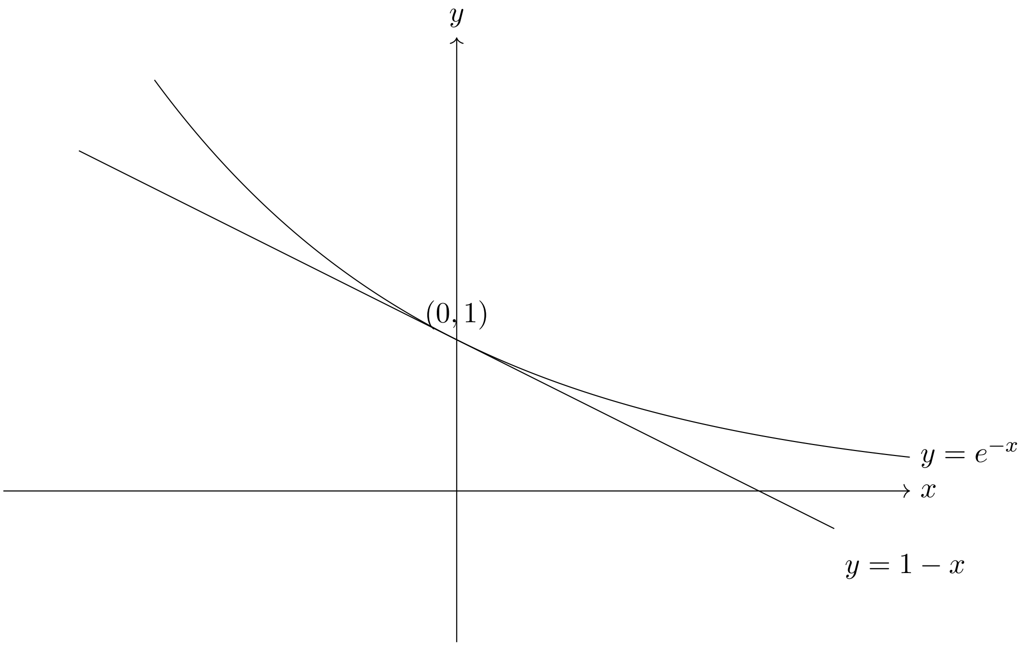
$$\begin{aligned} f'(t) &= \frac{(e^t - 1) - te^t}{(e^t-1)^2} \\ \lim_{t\to 0} f'(t) &= \lim_{t\to 0} \frac{(e^t - 1) - te^t}{(e^t-1)^2} \\ &= \lim_{t\to 0} \frac{t + \frac{t^2}{2} + o(t^3) - \left(t + t^2 + \frac{t^3}{2} + o(t^4)\right)}{(t + \frac{t^2}{2} + o(t^3))^2} \end{aligned}$$

$$= \lim_{t\to 0} \frac{-\frac{t^2}{2} + o(t^3)}{t^2 + o(t^3)}$$
$$= \lim_{t\to 0} \frac{-\frac{1}{2} + o(t)}{1 + o(t)}$$
$$\to -\frac{1}{2}$$

Claim $f(t) + \frac{1}{2}t$ is an even function. Proof: Consider $f(-t) - \frac{1}{2}t$, then

$$\begin{aligned} f(-t) - \frac{1}{2}t &= \frac{-t}{e^{-t} - 1} - \frac{1}{2}t \\ &= \frac{-te^t}{1 - e^t} - \frac{1}{2}t \\ &= \frac{t(1 - e^t) - t}{1 - e^t} - \frac{1}{2}t \\ &= t - \frac{t}{1 - e^t} - \frac{1}{2}t \\ &= \frac{t}{e^t - 1} + \frac{1}{2}t \end{aligned}$$

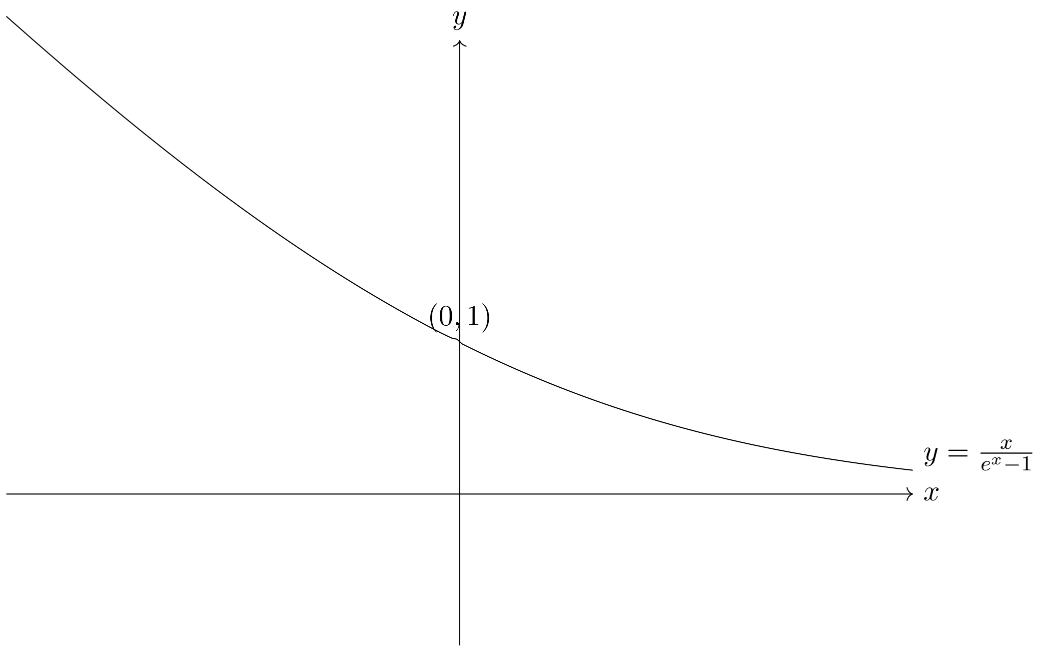
So it is even.



(ii)

Drawing the tangent to $y = e^{-x}$ at $(0, 1)$ we find that $e^{-t} \geq (1 - t)$ for all t, in particular, $e^t(1 - t) \leq 1$

$f'(t) = \frac{(e^t(1-t)-1}{(e^t-1)^2} \leq 0$ and $f'(t) = -\frac{1}{2}$ when $t = 0$



[Note: This is the exponential generating function for the Bernoulli numbers]

Question (2012 STEP II Q4)

In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}\,.$$

Hence show that $\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e}\,.$$

(ii) Show, using an expansion in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y}$ for $y > \frac{1}{2}$. Deduce that, for any positive integer n,

$$\mathrm{e} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\,.$$

(iii) Use parts (i) and (ii) to show that as $n \to \infty$

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e}\,.$$

(i) Since $k \geq 1$ we have $n^{k+1} > n^k$ and $(k+1) > k$, therefore $(k+1)n^{k+1} > kn^k \Rightarrow \frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}$

$$\begin{aligned}
\ln\left(1+\frac{1}{n}\right) &= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots \\
&= \frac{1}{n} - \underbrace{\left(\frac{1}{2n^2} - \frac{1}{3n^3}\right)}_{>0} - \underbrace{\left(\frac{1}{4n^4} - \frac{1}{5n^5}\right)}_{>0} - \cdot \\
&< \frac{1}{n}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & n\ln\left(1+\frac{1}{n}\right) < 1 \\
\Rightarrow \quad & \ln\left(\left(1+\frac{1}{n}\right)^n\right) < 1 \\
\Rightarrow \quad & \left(1+\frac{1}{n}\right)^n < e
\end{aligned}$$

(ii)

$$\begin{aligned}\ln\left(\frac{2y+1}{2y-1}\right) &= \ln\left(1+\frac{1}{2y}\right) - \ln\left(1-\frac{1}{2y}\right)\\ &= \frac{1}{2y} - \frac{1}{2(2y)^2} + \frac{1}{3(2y)^3} - \cdots - \left(-\frac{1}{2y} - \frac{1}{2(2y)^2} - \frac{1}{3(2y)^3} - \cdots\right)\\ &= \frac{1}{y} + \frac{2}{3(2y)^3} + \frac{2}{5(2y)^5}\\ &= \sum_{r=1}^{\infty} \frac{2}{(2r-1)(2y)^{2r-1}}\\ &> \frac{1}{y}\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad & \ln\left(1+\frac{1}{y-\frac{1}{2}}\right) > \frac{1}{y}\\ \Rightarrow \quad & \ln\left(1+\frac{1}{n}\right) > \frac{1}{n+\frac{1}{2}}\\ \Rightarrow \quad & (n+\tfrac{1}{2})\ln\left(1+\frac{1}{n}\right) > 1\\ \Rightarrow \quad & \ln\left(\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\right) > 1\\ \Rightarrow \quad & \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > e\end{aligned}$$

Since $\left(1+\frac{1}{n}\right)^n$ is both bounded above, and increasing, it must tend to some limit L.

$$\begin{aligned} & \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\\ \Rightarrow \quad & \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\\ \Rightarrow \quad & \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\end{aligned}$$

And therefore equality must hold.

Question (2012 STEP III Q4) **(i)** Show that

$$\sum_{n=1}^{\infty} \frac{n+1}{n!} = 2\mathrm{e} - 1$$

and

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} = 5\mathrm{e} - 1 .$$

Sum the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!}$.

(ii) Sum the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)}$, giving your answer in terms of natural logarithms.

(i)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n+1}{n!} &= \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)!} + \frac{1}{n!} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\
&= e + e - 1 \\
&= 2e - 1
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} &= \sum_{n=1}^{\infty} \frac{n(n-1) + 3n + 1}{n!} \\
&= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} + 3\sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\
&= 5e - 1
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!} &= \sum_{n=1}^{\infty} \frac{8n^3 - 12n^2 + 6n - 1}{n!} \\
&= \sum_{n=1}^{\infty} \frac{8n(n-1)(n-2) + 12n^2 - 10n - 1}{n!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{8n(n-1)(n-2)+12n(n-1)+2n-1}{n!} \\
&= 8e + 12e + 2e - (e-1) \\
&= 21e + 1
\end{aligned}$$

(ii)

$\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)} = \sum_{n=0}^{\infty} 2^{-n} + 2\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} - 5\sum_{n=0}^{\infty} \frac{2^{-n}}{n+2}$
$= 2 + 2\log 2 - 5\sum_{n=2}^{\infty} \frac{2^{-n+2}}{n}$
$= 2 + 2\log 2 - 5\left(2\log 2 - 2\right)$
$= 12 - 8\log 2$

Question (2013 STEP III Q2)

In this question, you may ignore questions of convergence. Let $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = 0$$

and prove that, for any positive integer n,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+3)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (n+1)^2\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0\,.$$

Hence obtain the Maclaurin series for $\dfrac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and for even powers of x. Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2\times 3^2}{7!} + \cdots + \frac{2^2\times 3^2\times\cdots\times n^2}{(2n+1)!} + \cdots\,.$$

$$\begin{aligned}
y &= \frac{\arcsin x}{\sqrt{1-x^2}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(1-x^2)^{-1/2}\cdot(1-x^2)^{1/2} - \arcsin x\cdot(-x)(1-x^2)^{-1/2}}{1-x^2} \\
&= \frac{1+xy}{1-x^2} \\
\Rightarrow \quad 0 &= (1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^{n+1}}: \quad 0 &= \left((1-x^2)y'\right)^{(n+1)} - (xy)^{(n+1)} \\
\Rightarrow \quad 0 &= (1-x^2)y^{(n+2)} + \binom{n+1}{1}(1-x^2)^{(1)}y^{(n+1)} + \binom{n+1}{2}(1-x^2)^{(2)}y^{(n)} - (xy^{(n+1)} + \binom{n+1}{1} \\
&= (1-x^2)y^{(n+2)} + \left((n+1)\cdot(-2x) - x\right)y^{(n+1)} + \left(\frac{(n+1)n}{2}\cdot(-2) - (n+1)\right)y^{(n)}
\end{aligned}$$

$$= (1-x^2)y^{(n+2)} - (2n+3)\,xy^{(n+1)} - ((n+1)n + (n+1))\,y^{(n)}$$
$$= (1-x^2)y^{(n+2)} - (2n+3)\,xy^{(n+1)} - (n+1)^2\,y^{(n)}$$

Since $y(0) = 0, y'(0) = 1$ we can look at the recursion: $y^{(n+2)} - (n+1)^2 y^{(n)}$ for larger terms, ie $y^{(2k)}(0) = 0$

$y^{(1)}(0) = 1, y^{(3)}(0) = (1+1)^2 \cdot 1 = 2^2, y^{(5)}(0) = (3+1)^2 y^{(3)} = 4^2 \cdot 2^2$ and $y^{(2k+1)}(0) = (2k)^2 \cdot (2k-2)^2 \cdots 2^2 \cdot 1^2 = 2^{2k} \cdot (k!)^2$. Therefore

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{2^{2k} \cdot (k!)^2}{(2k+1)!} x^{2k+1}$$

$$\Rightarrow \quad \frac{\arcsin \frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \sum_{k=0}^{\infty} \frac{2^{2k} \cdot (k!)^2}{(2k+1)!} 2^{-2k-1}$$
$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+1)!}$$
$$= \frac{1}{2}\left(1 + \frac{1}{3!} + \frac{2^2}{5!} + \cdots + \right)$$
$$\Rightarrow \quad S = 2\frac{2\frac{\pi}{6}}{\sqrt{3}} = \frac{2\pi}{3\sqrt{6}}$$

Question (2015 STEP II Q1) **(i)** By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)\,.$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2\,.$$

(i) Consider $f(x) = x - \ln(1+x)$, then $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ if $x > 0$.

Therefore $f(x)$ is strictly increasing on the positive reals. Since $f(0) = 0$ we must have $f(x) > 0$ for all positive x, ie $x - \ln(1+x)$ is positive for all positive x.

$$\sum_{k=1}^{n} \frac{1}{k} \underbrace{>}_{x > \ln(1+x)} \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right)$$
$$= \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right)$$

$$\begin{aligned} &= \sum_{k=1}^{n} (\ln(k+1) - \ln(k)) \\ &= \ln(n+1) - \ln 1 \\ &= \ln(n+1) \end{aligned}$$

(ii) Let $g(x) = x + \ln(1-x)$,then $g'(x) = 1 - \frac{1}{1-x} = \frac{-x}{1-x} < 0$ if $0 < x < 1$ and $g(0) = 0$. Therefore $g(x)$ is decreasing and hence negative on $0 < x < 1$, in particular $x < -\ln(1-x)$

$$\begin{aligned} \sum_{k=2}^{n} \frac{1}{k^2} &\underbrace{<}_{x<-\ln(1+x)} \sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right) \\ &= -\sum_{k=2}^{n} \ln\left(\frac{k^2-1}{k^2}\right) \\ &= \sum_{k=2}^{n} (2\ln k - \ln(k-1) - \ln(k+1)) \\ &= \ln n - \ln(n+1) - \ln 0 + \ln 2 \\ &= \ln 2 + \ln \frac{n}{n+1} \end{aligned}$$

as $n \to \infty$ we must have $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2} < \ln 2$ ie

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2$$

Question (2018 STEP II Q5)

In this question, you should ignore issues of convergence.

(i) Write down the binomial expansion, for $|x| < 1$, of $\dfrac{1}{1+x}$ and deduce that

$$\ln(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

for $|x| < 1$.

(ii) Write down the series expansion in powers of x of e^{-ax}. Use this expansion to show that

$$\int_0^{\infty} \frac{(1-\mathrm{e}^{-ax})\,\mathrm{e}^{-x}}{x}\,\mathrm{d}x = \ln(1+a) \qquad (|a| < 1).$$

(iii) Deduce the value of

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,\mathrm{d}x \qquad (|p| < 1,\ |q| < 1).$$

(i)

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \\
\Rightarrow \qquad \int_0^x \frac{1}{1+t}\mathrm{d}t &= \int_0^x \sum_{n=0}^{\infty} (-t)^n \mathrm{d}t \\
&= \left[\sum_{n=0}^{\infty} -\frac{(-t)^{n+1}}{n+1}\right]_0^x \\
\Rightarrow \qquad \ln(1+x) &= -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}
\end{aligned}$$

(ii)

$$\begin{aligned}
e^{-ax} &= \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} x^n \\
\Rightarrow \qquad \int_0^{\infty} \frac{1}{x}\left(1 - e^{-ax}\right) e^{-x}\mathrm{d}x &= \int_0^{\infty} \frac{1}{x}\left(-\sum_{n=1}^{\infty} \frac{(-a)^n}{n!} x^n\right) e^{-x}\mathrm{d}x \\
&= -\int_0^{\infty} \sum_{n=1}^{\infty} \frac{(-a)^n}{n!} x^{n-1} e^{-x}\mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{(-a)^n}{n!} \int_0^{\infty} x^{n-1} e^{-x}\mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{(-a)^n}{n!} (n-1)!
\end{aligned}$$

$$= -\sum_{n=1}^{\infty} \frac{(-a)^n}{n}$$
$$= \ln(1+a)$$

(iii)

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,\mathrm{d}x = \int_0^1 \frac{x^p(1-x^{q-p})}{\ln x}\,\mathrm{d}x$$

$e^{-u} = x, \mathrm{d}x = -e^{-u}\mathrm{d}u$:
$$= \int_{u=\infty}^{0} \frac{e^{-pu} - e^{-qu}}{-u}(-e^{-u})\mathrm{d}u$$
$$= \int_0^{\infty} \frac{e^{-u}(e^{-qu} - e^{-pu})}{u}\mathrm{d}u$$
$$= \int_0^{\infty} \frac{e^{-(1+q)u}(1 - e^{-(p-q)u})}{u}\mathrm{d}u$$

$v = (1+q)u, \mathrm{d}v = (1+q)\mathrm{d}u$:
$$= \int_0^{\infty} \frac{e^{-v}(1 - e^{-\left(\frac{p-q}{1+q}\right)v}}{v}\mathrm{d}v$$
$$= \ln\left(1 + \frac{p-q}{1+q}\right)$$
$$= \ln\left(\frac{1+p}{1+q}\right)$$

Question (2018 STEP III Q8)

In this question, you should ignore issues of convergence.

(i) Let

$$I = \int_0^1 \frac{f(x^{-1})}{1+x}\,\mathrm{d}x\,,$$

where $f(x)$ is a function for which the integral exists. Show that

$$I = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{f(y)}{y(1+y)}\,\mathrm{d}y$$

and deduce that, if $f(x) = f(x+1)$ for all x, then

$$I = \int_0^1 \frac{f(x)}{1+x}\,\mathrm{d}x\,.$$

(ii) The *fractional part*, $\{x\}$, of a real number x is defined to be $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example $\{3.2\} = 0.2$ and $\{3\} = 0$. Use the result of part (i) to evaluate

$$\int_0^1 \frac{\{x^{-1}\}}{1+x}\,\mathrm{d}x \text{ and } \int_0^1 \frac{\{2x^{-1}\}}{1+x}\,\mathrm{d}x\,.$$

(iii) (Bonus) Use the same method to evaluate

$$\int_0^1 \frac{x\{x^{-1}\}}{1-x^2}\,\mathrm{d}x\,.$$

(iv) (Bonus - harder) Use the same method to evaluate

$$\int_0^1 \frac{x^2\{x^{-1}\}}{1-x^2}\,\mathrm{d}x\,.$$

(i)

$$\begin{aligned}
I &= \int_0^1 \frac{f(x^{-1})}{1+x}\mathrm{d}x \\
u = x^{-1}, \mathrm{d}u = -x^{-2}\mathrm{d}x: \qquad &= \int_\infty^1 \frac{f(u)}{1+\frac{1}{u}}\frac{-1}{u^2}\mathrm{d}u \\
&= \int_1^\infty \frac{f(u)}{u(1+u)}\mathrm{d}u \\
&= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(u)}{u(u+1)}\mathrm{d}u \\
\text{if} f(x) = f(x+1)\,\forall x \qquad &= \sum_{n=1}^\infty \int_0^1 \frac{f(x+n)}{(x+n)(x+n+1)}\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \int_0^1 \frac{f(x)}{(x+n)(x+n+1)} \mathrm{d}x \\
&= \int_0^1 f(x) \left(\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)} \right) \mathrm{d}x \\
&= \int_0^1 f(x) \left(\sum_{n=1}^{\infty} \left(\frac{1}{x+n} - \frac{1}{x+n+1} \right) \right) \mathrm{d}x \\
&= \int_0^1 f(x) \left(\frac{1}{x+1} \right) \mathrm{d}x \\
&= \int_0^1 \frac{f(x)}{x+1} \mathrm{d}x
\end{aligned}$$

(ii) Since the fractional part is periodic with period 1, we can say

$$\begin{aligned}
\int_0^1 \frac{\{x^{-1}\}}{1+x} \mathrm{d}x &= \int_0^1 \frac{\{x\}}{x+1} \mathrm{d}x \\
&= \int_0^1 \frac{x}{x+1} \mathrm{d}x \\
&= \int_0^1 1 - \frac{1}{x+1} \mathrm{d}x \\
&= [x - \ln(1+x)]_0^1 \\
&= 1 - \ln 2
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{\{2x^{-1}\}}{1+x} \mathrm{d}x &= \int_0^1 \frac{\{2x\}}{x+1} \mathrm{d}x \\
&= \int_0^{1/2} \frac{2x}{x+1} \mathrm{d}x + \int_{1/2}^1 \frac{2x-1}{x+1} \mathrm{d}x \\
&= 2 \int_0^1 \frac{x}{x+1} \mathrm{d}x + \int_{1/2}^1 \frac{-1}{x+1} \mathrm{d}x \\
&= 2 - 2\ln 2 - \left(\ln 2 - \ln \tfrac{3}{2} \right) \\
&= 2 - 4\ln 2 + \ln 3 \\
&= 2 + \ln \tfrac{3}{16}
\end{aligned}$$

(iii)

$$\int_0^1 \frac{x\{x^{-1}\}}{1-x^2} \mathrm{d}x = \frac{1}{2} \left(\int_0^1 \frac{\{x^{-1}\}}{1-x} - \frac{\{x^{-1}\}}{1+x} \mathrm{d}x \right)$$

Consider for f periodic with period 1

$$\int_0^1 \frac{f(x^{-1})}{1-x} \mathrm{d}x = \int_1^{\infty} \frac{f(u)}{u(u-1)} \mathrm{d}u$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{f(u)}{u(u-1)} \mathrm{d}u \\
&= \sum_{n=1}^{\infty} \int_0^1 \frac{f(u)}{(u+n)(u+n-1)} \mathrm{d}u \\
&= \int_0^1 \sum_{n=1}^{\infty} \frac{f(u)}{(u+n)(u+n-1)} \mathrm{d}u \\
&= \int_0^1 f(u) \sum_{n=1}^{\infty} \left(\frac{1}{u+n-1} - \frac{1}{u+n} \right) \mathrm{d}u \\
&= \int_0^1 \frac{f(u)}{u} \mathrm{d}u
\end{aligned}
$$

So we have

$$
\begin{aligned}
\int_0^1 \frac{x\{x^{-1}\}}{1-x^2} \mathrm{d}x &= \frac{1}{2} \left(\int_0^1 \frac{\{x^{-1}\}}{1-x} - \frac{\{x^{-1}\}}{1+x} \mathrm{d}x \right) \\
&= \frac{1}{2} \int_0^1 \frac{\{x\}}{x} \mathrm{d}x - \frac{1}{2}(1 - \ln 2) \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \ln 2 \\
&= \frac{1}{2} \ln 2
\end{aligned}
$$

(i) Notice that $f(x) = x - \tanh x$ has $f'(x) = 1 - \mathrm{sech}^2 x = \tanh^2 x > 0$ so $f(x)$ is strictly increasing on $(0, \infty)$ and $f(0) = 0$ therefore $f(x)$ is positive for all x positive

(ii) Let $f(x) = x \sinh x - 2 \cosh x + 2$ then $f'(x) = \sinh x + x \cosh x - 2 \sinh x = x \cosh x - \sinh x = \cosh x(x - \tanh x) > 0$ by the first part. $f(0) = 0$ so $f(x)$ is positive for all x positive.

(iii) Let $f(x) = 2x \cosh 2x - 3 \sinh 2x + 4x$ then

$$
\begin{aligned}
f'(x) &= 2 \cosh 2x + 4x \sinh 2x - 6 \cosh 2x + 4 \\
&= 4(x \sinh 2x - \cosh 2x + 1) \\
&= 4(x 2 \cosh x \sinh x - 2 \cosh^2 x) \\
&= 8 \cosh^2 x(x - \tanh x)
\end{aligned}
$$

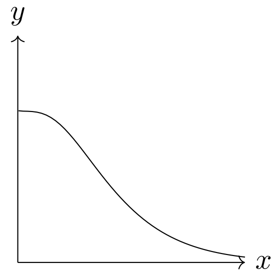
Which is always positive when x ¿ 0, $f(0) = 0$ so $f(x) > 0$ for all positive x.

Let $f(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}$ then

$$
\begin{aligned}
f'(x) &= \frac{(\cosh x)^{\frac{1}{3}} \sinh x + \frac{1}{3} x \cosh^{-\frac{2}{3}} x \sinh^2 x - x(\cosh x)^{\frac{1}{3}} \cosh x}{\sinh^2 x} \\
&= \frac{\cosh x \sinh x + \frac{1}{3} x \sinh^2 x - x \cosh^2 x}{\cosh x^{\frac{2}{3}} x \sinh^2 x}
\end{aligned}
$$

$$= \frac{3\cosh x \sinh x + x(\sinh^2 x - 3\cosh^2 x)}{3\cosh x^{\frac{2}{3}} x \sinh^2 x}$$
$$= \frac{\frac{3}{2}\sinh 2x + x(-2\cosh 2x - 2)}{3\cosh x^{\frac{2}{3}} x \sinh^2 x}$$
$$= \frac{3\sinh 2x - 4x\cosh 2x - 4x}{6\cosh x^{\frac{2}{3}} x \sinh^2 x}$$

which from the earlier part is always negative.



Question (1989 STEP II Q3)

The real numbers x and y are related to the real numbers u and v by

$$2(u + \mathrm{i}v) = \mathrm{e}^{x+\mathrm{i}y} - \mathrm{e}^{-x-\mathrm{i}y}.$$

Show that the line in the x-y plane given by $x = a$, where a is a positive constant, corresponds to the ellipse

$$\left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2 = 1$$

in the u-v plane. Show also that the line given by $y = b$, where b is a constant and $0 < \sin b < 1$, corresponds to one branch of a hyperbola in the u-v plane. Write down the u and v coordinates of one point of intersection of the ellipse and hyperbola branch, and show that the curves intersect at right-angles at this point. Make a sketch of the u-v plane showing the ellipse, the hyperbola branch and the line segments corresponding to:

(i) $x = 0$;

(ii) $y = \frac{1}{2}\pi, \quad 0 \leqslant x \leqslant a.$

$$\begin{aligned} 2(u+iv) &= e^{a+iy} - e^{-a-iy} \\ &= (e^a \cos y - e^{-a}\cos y) + (e^a \sin y + e^{-a}\sin y)i \\ &= 2\sinh a \cos y + 2\cosh a \sin y i \end{aligned}$$

$$\Rightarrow \quad \frac{u}{\sinh a} = \cos y$$
$$\frac{v}{\cosh a} = \sin y$$
$$\Rightarrow \quad 1 = \left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2$$

$$\begin{aligned} 2(u+iv) &= e^{x+ib} - e^{-x-ib} \\ &= 2\sinh x \cos b + 2\cosh x \sin b i \\ \Rightarrow \qquad \frac{u}{\cos b} &= \sinh x \\ \frac{v}{\sin b} &= \cosh x \\ \Rightarrow \qquad 1 &= \left(\frac{v}{\sin b}\right)^2 - \left(\frac{u}{\cos b}\right)^2 \end{aligned}$$

Therefore all the points lie of a hyperbola, and since $\frac{v}{\sin b} > 0 \Rightarrow v > 0$ it's one branch of the hyperbola. (And all points on it are reachable as x varies from $-\infty < x < \infty$.

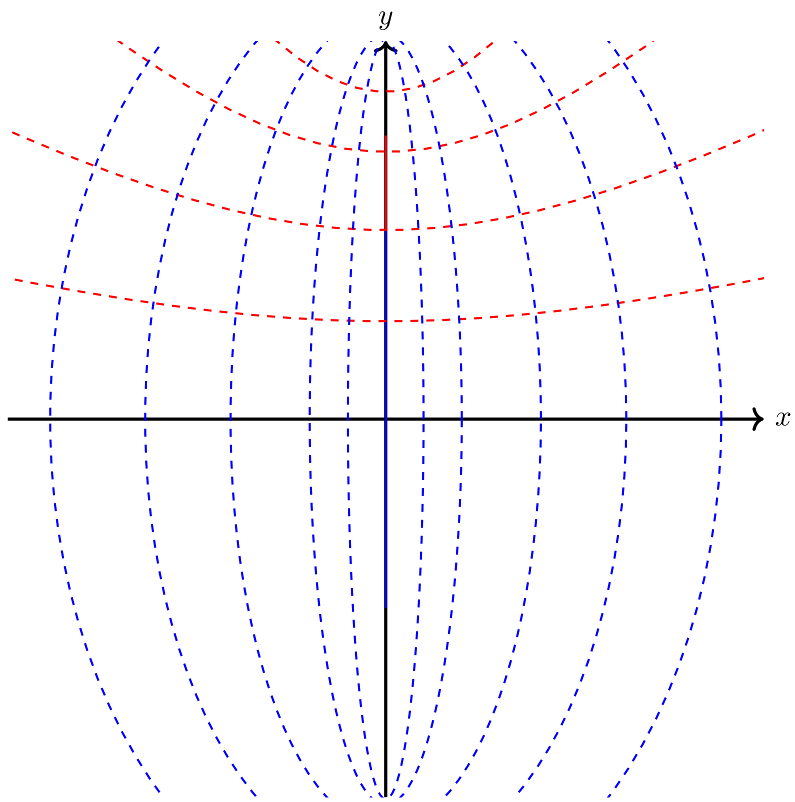
$$\begin{aligned} 2(u+iv) &= e^{a+ib} - e^{-a-ib} \\ &= 2\sinh a \cos b + 2\cosh a \sin b i \end{aligned}$$

so we can take $u = \sinh a \cos b, v = \cosh a \sin b$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} \qquad 0 &= \frac{2u}{\sinh^2 a} + \frac{2v}{\cosh^2 a}\frac{\mathrm{d}v}{\mathrm{d}u} \\ \Rightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}u} &= -\frac{u}{v}\coth^2 a \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}u}\Big|_{(u,v)} &= -\frac{\sinh a \cos b}{\cosh a \sin b}\coth^2 a \\ &= -\cot b \coth a \\ \frac{\mathrm{d}}{\mathrm{d}u} \qquad 0 &= \frac{2v}{\sin^2 b}\frac{\mathrm{d}v}{\mathrm{d}u} - \frac{2u}{\cos^2 b} \\ \Rightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{u}{v}\tan^2 b \\ \frac{\mathrm{d}v}{\mathrm{d}u}\Big|_{(u,v)} &= \frac{\sinh a \cos b}{\cosh a \sin b}\tan^2 b \\ &= \tanh a \tan b \end{aligned}$$

Therefore they are negative reciprocals and hence perpendicular.



Question (1989 STEP III Q5)

Given that $y = \cosh(n\cosh^{-1} x)$, for $x \geqslant 1$, prove that

$$y = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}.$$

Explain why, when $n = 2k+1$ and $k \in \mathbb{Z}^+$, y can also be expressed as the polynomial

$$a_0 x + a_1 x^3 + a_2 x^5 + \cdots + a_k x^{2k+1}.$$

Find a_0, and show that

(i) $a_1 = (-1)^{k-1} 2k(k+1)(2k+1)/3$;

(ii) $a_2 = (-1)^k 2(k-1)k(k+2)(2k+1)/15$.

Find also the value of $\displaystyle\sum_{r=0}^{k} a_r$.

Recall, $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$

$$\begin{aligned}
\cosh(n\cosh^{-1} x) &= \frac{1}{2}\left(\exp(n\cosh^{-1} x) + \exp(-n\cosh^{-1} x)\right) \\
&= \frac{1}{2}\left((x+\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^{-n}\right) \\
&= \frac{1}{2}\left((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right)
\end{aligned}$$

When $n = 2k+1$

$$\begin{aligned}
\cosh(n\cosh^{-1}x) &= \frac{1}{2}\left((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right) \\
&= \frac{1}{2}\left(\sum_{i=0}^{2k+1}\binom{2k+1}{i}x^{2k+1-i}\left((\sqrt{x^2-1}^{\,i} + (-\sqrt{x^2-1})^i\right)\right) \\
&= \sum_{i=0}^{k}\binom{2k+1}{2i}x^{2k+1-2i}(x^2-1)^i \\
&= \sum_{i=0}^{k}\binom{2k+1}{2i}x^{2(k-i)+1}(x^2-1)^i
\end{aligned}$$

Which is clearly a polynomial with only odd degree terms.

$$\begin{aligned}
a_0 &= \frac{\mathrm{d}y}{\mathrm{d}x}|_{x=0} \\
&= \sum_{i=0}^{k}\binom{2k+1}{2i}\left((2(k-i)+1)x^{2(k-i)}(x^2-1)^i + 2i\cdot x^{2(k-i)+2}(x^2-1)\right) \\
&= \binom{2k+1}{2k}(-1)^k \\
&= (-1)^k(2k+1)
\end{aligned}$$

(i)

$$\begin{aligned}
a_1 &= \binom{2k+1}{2k}\binom{k}{1}(-1)^{k-1} + \binom{2k+1}{2(k-1)}(-1)^{k-1} \\
&= (-1)^{k-1}\cdot((2k+1)k + \frac{(2k+1)\cdot 2k\cdot(2k-1)}{3!}) \\
&= (-1)^{k-1}(2k+1)k\frac{3+2k-1}{3} \\
&= (-1)^{k-1}2(2k+1)k(k+1)
\end{aligned}$$

(ii)

$$\begin{aligned}
a_2 &= \binom{2k+1}{2k}\binom{k}{2}(-1)^{k-2} + \binom{2k+1}{2(k-1)}\binom{k-1}{1}(-1)^{k-2} + \binom{2k+1}{2(k-2)}(-1)^{k-2} \\
&= \binom{2k+1}{1}\binom{k}{2}(-1)^{k-2} + \binom{2k+1}{3}\binom{k-1}{1}(-1)^{k-2} + \binom{2k+1}{5}(-1)^{k-2} \\
&= (-1)^k\left(\binom{2k+1}{1}\frac{k(k-1)}{2} + \binom{2k+1}{3}(k-1) + \binom{2k+1}{5}\right) \\
&= (-1)^k\left(\frac{(2k+1)k(k-1)}{2} + \frac{(2k+1)k(2k-1)}{3} + \frac{(2k+1)k(2k-1)(k-1)(2k-3)}{5\cdot 2\cdot 3}\right) \\
&= (-1)^k(2k+1)k\frac{1}{30}\left(15(k-1) + 10(2k-1) + (2k-1)(k-1)(2k-3)\right)
\end{aligned}$$

$$\begin{aligned}\sum_{r=0}^{k} a_k &= \frac{1}{2}\left((1+\sqrt{1^2-1})^n + (1-\sqrt{1^2-1})^n\right) \\ &= 1\end{aligned}$$

Question (1990 STEP III Q9)

The real variables θ and u are related by the equation $\tan\theta = \sinh u$ and $0 \leqslant \theta < \frac{1}{2}\pi$. Let $v = \text{sech}u$. Prove that

(i) $v = \cos\theta$;

(ii) $\dfrac{\mathrm{d}\theta}{\mathrm{d}u} = v$;

(iii) $\sin 2\theta = -2\dfrac{\mathrm{d}v}{\mathrm{d}u}$ and $\cos 2\theta = -\cosh u\dfrac{\mathrm{d}^2v}{\mathrm{d}u^2}$;

(iv) $\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\dfrac{\mathrm{d}^2v}{\mathrm{d}\theta^2} + \dfrac{\mathrm{d}v}{\mathrm{d}\theta}\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left(\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = 0.$

(i)

$$\begin{aligned} v &= \text{sech}u \\ &= \frac{1}{\text{cosh}u} \\ &= \frac{1}{\sqrt{1+\text{sinh}^2 u}} && (u > 0) \\ &= \frac{1}{\sqrt{1+\tan^2\theta}} \\ &= \frac{1}{\sqrt{\sec^2\theta}} \\ &= \cos\theta && (0 < \theta < \tfrac{\pi}{2}) \end{aligned}$$

(ii)

$$\begin{aligned} && \tan\theta &= \text{sinh}u \\ \underbrace{\Rightarrow}_{\frac{\mathrm{d}}{\mathrm{d}u}} && \sec^2\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}u} &= \cosh u \\ \Rightarrow && \frac{\mathrm{d}\theta}{\mathrm{d}u} &= \cosh u \cdot \cos^2\theta \\ && &= \frac{1}{v}\cdot v^2 \\ && &= v \end{aligned}$$

(iii)

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\sin\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} \\ &= -2\frac{\mathrm{d}v}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} && (\cos\theta = v) \\ &= -2\frac{\mathrm{d}v}{\mathrm{d}u}\end{aligned}$$

$$\begin{aligned} & & \sin 2\theta &= -2\frac{\mathrm{d}v}{\mathrm{d}u} \\ &\underbrace{\Rightarrow}_{\frac{\mathrm{d}}{\mathrm{d}u}} & 2\cos 2\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}u} &= -2\frac{\mathrm{d}^2v}{\mathrm{d}u^2} \\ &\Rightarrow & \cos 2\theta &= -\frac{\mathrm{d}^2v}{\mathrm{d}u^2}\frac{1}{v} \\ & & &= -\frac{\mathrm{d}^2v}{\mathrm{d}u^2}\cosh u\end{aligned}$$

(iv)

$$\begin{aligned} & & \frac{\mathrm{d}u}{\mathrm{d}\theta} &= \frac{1}{v} \\ &\Rightarrow & \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} &= -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}\theta} \\ & & &= \frac{1}{v^2}\sin\theta \\ & & \frac{\mathrm{d}v}{\mathrm{d}\theta} &= -\sin\theta \\ &\Rightarrow & \frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} &= -\cos\theta \\ & & &= -v\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2}+\frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 &= \frac{1}{v}\cdot(-v)+(-\sin\theta)\cdot\left(\frac{1}{v^2}\sin\theta\right)+\frac{1}{v^2} \\ &= -1+\frac{1-\sin^2\theta}{v^2} \\ &= -1+\frac{\cos^2\theta}{v^2} \\ &= -1+1 \\ &= 0\end{aligned}$$

Question (1991 STEP II Q8)

Solve the quadratic equation $u^2 + 2u\sinh x - 1 = 0$, giving u in terms of x. Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

which satisfies $y = 0$ and $y' > 0$ at $x = 0$. Find the solution of the differential equation

$$\sinh x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x = 0$$

which satisfies $y = 0$ at $x = 0$.

$$\begin{aligned} 0 &= u^2 + 2u\sinh x - 1 \\ &= u^2 + u(e^x - e^{-x}) - e^x e^{-x} \\ &= (u - e^{-x})(u + e^x) \\ \Rightarrow \quad u &= e^{-x}, -e^x \end{aligned}$$

$$\begin{aligned} 0 &= \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 \\ \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= e^{-x}, -e^x \\ \Rightarrow \quad y &= -e^{-x} + C, -e^x + C \\ y(0) = 0: \quad C &= 1 \text{ both cases} \\ y'(0) > 0: \quad y &= 1 - e^{-x} \end{aligned}$$

$$\begin{aligned} 0 &= \sinh x u^2 + 2u - \sinh x \\ \Rightarrow \quad u &= \frac{-2 \pm \sqrt{4 + 4\sinh^2 x}}{2\sinh x} \\ &= \frac{-1 \pm \cosh x}{\sinh x} = -\text{cosech } x \pm \coth x \end{aligned}$$

$$\begin{aligned} 0 &= \sinh x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x \\ \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= -\text{cosech } x \pm \coth x \\ \Rightarrow \quad y &= -\ln\left(\tanh\frac{x}{2}\right) \pm \ln\sinh x + C \end{aligned}$$

For $x \to 0$ to be defined, we need $+$, so

$$\begin{aligned} y &= \ln\left(\frac{\sinh x}{\tanh\frac{x}{2}}\right) + C \\ y &= \ln\left(\frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\tanh\frac{x}{2}}\right) + C \end{aligned}$$

$$
\begin{aligned}
& & &= \ln\left(2\cosh^2 x\right) + C \\
y(0) = 0: & & 0 &= \ln 2 + C \\
\Rightarrow & & y &= \ln(2\cosh^2 x) - \ln 2 \\
& & y &= 2\ln(\cosh x)
\end{aligned}
$$

Question (1991 STEP III Q6)
The transformation T from $\binom{x}{y}$ to $\binom{x'}{y'}$ in two-dimensional space is given by

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
$$

where u is a positive real constant. Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves. A line, not necessary through the origin, which has gradient $\tanh v$ transforms under T into a line with gradient $\tanh v'$. Show that $v' = v + u$. The lines ℓ_1 and ℓ_2 with gradients $\tanh v_1$ and $\tanh v_2$ transform under T into lines with gradients $\tanh v_1'$ and $\tanh v_2'$ respectively. Find the relation satisfied by v_1 and v_2 that is the necessary and sufficient for ℓ_1 and ℓ_2 to intersect at the same angle as their transforms. In the case when ℓ_1 and ℓ_2 meet at the origin, illustrate in a diagram the relation between ℓ_1, ℓ_2 and their transforms.

None

Question (1992 STEP III Q1) **(i)** Given that

$$
\mathrm{f}(x) = \ln(1 + \mathrm{e}^x),
$$

prove that $\ln[\mathrm{f}'(x)] = x - \mathrm{f}(x)$ and that $\mathrm{f}''(x) = \mathrm{f}'(x) - [\mathrm{f}'(x)]^2$. Hence, or otherwise, expand $\mathrm{f}(x)$ as a series in powers of x up to the term in x^4.

(ii) Given that

$$
\mathrm{g}(x) = \frac{1}{\sinh x \cosh 2x},
$$

explain why $\mathrm{g}(x)$ can not be expanded as a series of non-negative powers of x but that $x\mathrm{g}(x)$ can be so expanded. Explain also why this latter expansion will consist of even powers of x only. Expand $x\mathrm{g}(x)$ as a series as far as the term in x^4.

(i)

$$
\begin{aligned}
& & f(x) &= \ln(1 + e^x) \\
& & f'(x) &= \frac{1}{1 + e^x} \cdot e^x \\
& & &= \frac{e^x}{1 + e^x} \\
\Rightarrow & & \ln[f'(x)] &= x - \ln(1 + e^x) \\
& & &= x - f(x)
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow \qquad \frac{f''(x)}{f'(x)} &= 1 - f'(x) \\
\Rightarrow \qquad f''(x) &= f'(x) - [f'(x)]^2 \\
f'''(x) &= f''(x) - 2f'(x)f''(x) \\
f^{(4)}(x) &= f'''(x) - 2[f''(x)]^2 - 2f'(x)f'''(x)
\end{aligned}$$

$$\begin{aligned}
f(0) &= \ln 2 \\
f'(0) &= \tfrac{1}{2} \\
f''(0) &= \tfrac{1}{2} - \tfrac{1}{4} \\
&= \tfrac{1}{4} \\
f'''(0) &= \tfrac{1}{4} - 2\tfrac{1}{2}\tfrac{1}{4} \\
&= 0 \\
f^{(4)}(0) &= -2 \cdot \tfrac{1}{16} \\
&= -\frac{1}{8}
\end{aligned}$$

Therefore $f(x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{8 \cdot 4!}x^4 + O(x^5)$

(ii) As $x \to 0$, $g(x) \to \infty$ therefore there can be no power series about 0. But as $x \to 0, xg(x) \not\to \infty$ as $\frac{x}{\sinh x}$ is well behaved.

We can also notice that $xg(x)$ is an even function, since $\cosh x$ is even and $\frac{x}{\sinh x}$ is even, therefore the power series will consist of even powers of x

$$\begin{aligned}
\lim_{x \to 0} \frac{x}{\sinh x \cosh 2x} &= \lim_{x \to 0} \frac{x}{\sinh x} \cdot \lim_{x \to 0} \frac{1}{\cosh 2x} \\
&= 1
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{x}{\sinh x \cosh 2x} &= \frac{4x}{(e^x - e^{-x})(e^{2x} + e^{-2x})} \\
&= \frac{4x}{(2x + \frac{x^3}{3} + \cdots)(2 + 4x^2 + \frac{4}{3}x^4 + \cdots)} \\
&= \frac{1}{1 + \frac{x^2}{6} + \frac{x^4}{5!} + \cdots} \frac{1}{1 + 2x^2 + \frac{2}{3}x^4 + \cdots} \\
&= \left(1 - (\frac{x^2}{6} + \frac{x^4}{5!}) + (\frac{x^2}{6})^2 + O(x^6)\right)\left(1 - (2x^2 + \frac{2}{3}x^4) + (2x^2)^2 + O(x^6)\right) \\
&= \left(1 - \frac{1}{6}x^2 + \frac{7}{360}x^4 + O(x^6)\right)\left(1 - 2x^2 + \frac{10}{3}x^4 + O(x^6)\right) \\
&= 1 - \frac{13}{6}x^2 + \frac{1327}{360}x^4 + O(x^6)
\end{aligned}$$

Question (1993 STEP III Q7)

The real numbers x and y satisfy the simultaneous equations

$$\sinh(2x) = \cosh y \qquad \text{and} \qquad \sinh(2y) = 2\cosh x.$$

Show that $\sinh^2 y$ is a root of the equation

$$4t^3 + 4t^2 - 4t - 1 = 0$$

and demonstrate that this gives at most one valid solution for y. Show that the relevant value of t lies between 0.7 and 0.8, and use an iterative process to find t to 6 decimal places. Find y and hence find x, checking your answers and stating the final answers to four decimal places.

Let $t = \sinh^2 y$, then

$$\begin{aligned} \sinh(2x) &= \cosh y && (1) \\ \sinh(2y) &= 2\cosh x && (2) \end{aligned}$$

$$\begin{aligned}
\cosh(2x) &= 2\cosh^2 x - 1 \\
(2): \quad &= \frac{1}{2}\sinh^2(2y) - 1 \\
1 &= \left(\frac{1}{2}\sinh^2(2y) - 1\right)^2 - \cosh^2 y \\
&= \frac{1}{4}\sinh^4(2y) - \sinh^2(2y) + 1 - \cosh^2 y \\
\Rightarrow \quad 0 &= \frac{1}{4}(\cosh^2(2y) - 1)^2 - (\cosh^2(2y) - 1) - \cosh^2 y \\
&= \frac{1}{4}\left(\left(1 + 2\sinh^2 y\right)^2 - 1\right)^2 - \left(\left(1 + 2\sinh^2 y\right)^2 - 1\right) - (1 + \sinh^2 y) \\
&= \frac{1}{4}\left(1 + 4t + 4t^2 - 1\right)^2 - \left(1 + 4t + 4t^2 - 1\right) - (1 + t) \\
&= \frac{1}{4}(4t + 4t^2)^2 - (4t + 4t^2) - 1 - t \\
&= 4(t + t^2)^2 - 4t^2 - 5t - 1 \\
&= 4t^4 + 8t^3 + 4t^2 - 4t^2 - 5t - 1 \\
&= 4t^4 + 8t^3 - 5t - 1 \\
&= (t + 1)(4t^3 + 4t^2 - 4t - 1)
\end{aligned}$$

Since $\sinh^2 y$ is positive, we must be a root of the second cubic.

Let $f(t) = 4t^3 + 4t^2 - 4t - 1$, then $f(0) = -1$ and $f'(t) = 12t^2 + 8t - 4 = 4(t+1)(3t-1)$, so we have turning points at -1 and $\frac{1}{3}$. Since $f(-1) = 3 > 0$ and $f(0) < 0$ we must have exactly one root larger than zero. Therefore there is a unique root.

$f(0.7) = -0.468 < 0$ $f(0.8) = 0.408 > 0$

since f is continuous and changes sign, the root must fall in the interval $(0.7, 0.8)$.

Let $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$, and $t_0 = 0.75$, then

$$t_0 = 0.75$$

$$\begin{aligned}
t_1 &= 0.7571428571 \\
t_2 &= 0.7570684728 \\
t_3 &= 0.7570684647
\end{aligned}$$

So $t \approx 0.757068$, $\sinh y \approx 0.870097$, $y \approx 0.786474$, $x \approx 0.546965$

Question (1996 STEP III Q1)

Define $\cosh x$ and $\sinh x$ in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \tfrac{1}{4}\cosh 4x + \tfrac{3}{4}.$$

Find $a_0, a_1, \dots, a_n$ in terms of n such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx.$$

Hence, or otherwise, find expressions for $\cosh^{2m} x - \sinh^{2m} x$ and $\cosh^{2m} x + \sinh^{2m} x$, in terms of $\cosh kx$, where $k = 0, \dots, 2m$.

$$\begin{aligned}
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
\sinh x &= \frac{1}{2}(e^x - e^{-x})
\end{aligned}$$

$$\begin{aligned}
\cosh^4 x - \sinh^4 x &= (\cosh^2 x - \sinh^2 x)(\cosh^2 x + \sinh^2 x) \\
&= \left(\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\right)(\cosh^2 x + \sinh^2 x) \\
&= (\cosh^2 x + \sinh^2 x) \\
&= \left(\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) + \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\right) \\
&= \frac{1}{4}\left(2e^{2x} + 2e^{-2x}\right) \\
&= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) \\
&= \cosh 2x
\end{aligned}$$

$$\begin{aligned}
\cosh^4 x + \sinh^4 x &= \frac{1}{2^4}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right) + \frac{1}{2^4}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right) \\
&= \frac{1}{8}(e^{4x} + e^{-4x}) + \frac{3}{4} \\
&= \frac{1}{4}\cosh 4x + \frac{3}{4}
\end{aligned}$$

$$\cosh^n x = \frac{1}{2^n}\left(e^x + e^{-x}\right)^n$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{kx}e^{-(n-k)x}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{2kx-nx}$$

$$= \frac{1}{2^n}\left(\binom{n}{n}\left(e^{nx}+e^{-nx}\right)+\binom{n}{n-1}\left(e^{(n-2)x}+e^{-(n-2)x}\right)+\cdots+\binom{n}{n-k}\left(e^{(n-2k)x}+e^{-(n-2k)x}\right)\right.$$

$$= \frac{1}{2^{n-1}}\cosh nx+\frac{1}{2^{n-1}}\binom{n}{n-1}\cosh(n-2)x+\cdots+\frac{1}{2^{n-1}}\binom{n}{n-k}\cosh(n-2k)x+\cdots$$

ie

$$\cosh^{2m}x = \frac{1}{2^{2m-1}}\cosh 2mx+\frac{2m}{2^{2m-1}}\cosh(2(m-1)x)+\cdots+\frac{1}{2^{2m-1}}\binom{2m}{k}\cosh(2(m-k)x)+\cdots$$

$$\sinh^{2m}x = \frac{1}{2^{2m-1}}\cosh 2mx-\frac{2m}{2^{2m-1}}\cosh(2(m-1)x)+\cdots+(-1)^k\frac{1}{2^{2m-1}}\binom{2m}{k}\cosh(2(m-k)$$

$$\cosh^{2m}x-\sinh^{2m}x = \frac{m}{2^{2m-3}}\cosh(2(m-1)x)+\cdots+\frac{1}{2^{2m-2}}\binom{2m}{2k+1}\cosh(2(m-2k-1)x)+\cdots$$

$$\cosh^{2m}x+\sinh^{2m}x = \frac{1}{2^{2m-2}}\cosh(2mx)+\cdots+\frac{1}{2^{2m-2}}\binom{2m}{2k}\cosh(2(m-2k)x)+\cdots$$

Question (2006 STEP III Q7) **(i)** Solve the equation $u^2+2u\sinh x-1=0$ giving u in terms of x. Find the solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}\sinh x-1=0$$

that satisfies $y=0$ and $\frac{dy}{dx}>0$ at $x=0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}-\sinh y=0$$

that satisfies $y=0$ at $x=0$, expressing your solution in the form $\cosh y=f(x)$. Show that the asymptotes to the solution curve are $y=\pm(-x+\ln 4)$.

Question (2007 STEP III Q5)

Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth\theta = x$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cosh\theta}{r} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\cosh 2\theta}{r^2},$$

and find an expression in terms of r and θ for $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Find, with proof, a similar formula for $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of r and θ.

$$\begin{aligned}
y &= \ln(x^2 - 1) \\
r &= \sqrt{x^2 - 1} \\
\coth\theta &= x \\
r &= \sqrt{\coth^2\theta - 1} = \sqrt{\mathrm{cosech}^2\theta} = \mathrm{cosech}\theta \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x}{x^2 - 1} \\
&= \frac{2\coth\theta}{r^2} \\
&= \frac{2\cosh\theta}{\sinh\theta \cdot r \cdot \mathrm{cosech}\theta} \\
&= \frac{2\cosh\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2(x^2 - 1) - 4x^2}{(x^2 - 1)^2} \\
&= \frac{-2(1 + x^2)}{r^2 \mathrm{cosech}^2 r} \\
&= -\frac{2(1 + \coth^2\theta)\sinh^2\theta}{r^2} \\
&= -\frac{2(\sinh^2\theta + \cosh^2\theta)}{r^2} \\
&= -\frac{2\cosh 2\theta}{r^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \frac{-4x(x^2 - 1)^2 - (-2x^2 - 2) \cdot 2(x^2 - 1) \cdot 2x}{(x^2 - 1)^4} \\
&= \frac{-4x(x^2 - 1) + 8x(x^2 + 1)}{(x^2 - 1)^3} \\
&= \frac{4x^3 + 12x}{(x^2 - 1)^3} \\
&= \frac{\sinh^3\theta(4\coth^3\theta + 12\coth\theta)}{r^3} \\
&= \frac{4\cosh^3\theta + 12\cosh\theta\sinh^2\theta}{r^3}
\end{aligned}$$

$$= \frac{4\cosh 3\theta}{r^3}$$

Claim: $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n+1}\frac{2(n-1)!\cosh n\theta}{r^n}$ Proof: By induction. Base cases already proven

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}x} &= \frac{x}{\sqrt{x^2-1}} = \frac{\coth\theta}{\mathrm{cosech}\theta} = \cosh\theta \\
\frac{\mathrm{d}\theta}{\mathrm{d}x} &= -\sinh^2\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} &= (-1)^{n+1}(n-1)!\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\cosh n\theta}{r^n}\right) \\
&= (-1)^{n+1}\frac{2n\sinh n\theta\cdot r^n\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x} - 2\cosh n\theta\cdot nr^{n-1}\frac{\mathrm{d}r}{\mathrm{d}x}}{r^{2n}} \\
&= (-1)^{n+2}\frac{2n(\cosh n\theta\cosh\theta + r\sinh n\theta\sinh^2\theta)}{r^{n+1}} \\
&= (-1)^{n+2}n!\frac{2\cosh(n+1)\theta}{r^{n+1}}
\end{aligned}$$

We can think of this as $\ln(x^2-1) = \ln(x+1)+\ln(x-1)$ and also note $x\pm 1 = \coth\theta\pm 1 = \frac{\cosh\theta\pm\sinh\theta}{\sinh\theta} = \frac{e^{\pm\theta}}{\sinh\theta}$

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\ln(x^2-1) &= (n-1)!(-1)^{n-1}\left(\frac{1}{(x+1)^n}+\frac{1}{(x-1)^n}\right) \\
&= (-1)^{n-1}(n-1)!\left(\frac{\sinh^n\theta}{e^{n\theta}}+\frac{\sinh^n\theta}{e^{-n\theta}}\right) \\
&= (-1)^{n-1}(n-1)!2\cosh n\theta\cdot\sinh^n\theta \\
&= (-1)^{n-1}(n-1)!\frac{2\cosh n\theta}{r^n}
\end{aligned}$$

Question (2014 STEP III Q6)

Starting from the result that

$$\dot{h}(t) > 0 \text{ for } 0 < t < x \implies \int_0^x \dot{h}(t)t > 0\,,$$

show that, if $\dot{f}''(t) > 0$ for $0 < t < x_0$ and $\dot{f}(0) = \dot{f}'(0) = 0$, then $\dot{f}(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x\cosh x < 1\,.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}\,.$$

None

Question (2016 STEP III Q6)

Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = x$ and $y = a\tanh x + b$, where $a > 0$.

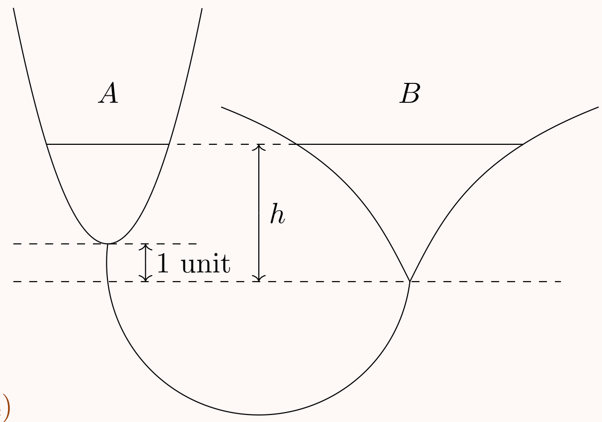
(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm\left(\frac{1}{\sqrt{b^2-a^2}}\right) - artanh\,\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.



Question (1987 STEP III Q4)

Two funnels A and B have surfaces formed by rotating the curves $y = x^2$ and $y = 2\sinh^{-1} x$ $(x > 0)$ above the y-axis. The bottom of B is one unit lower than the bottom of A and they are connected by a thin rubber tube with a tap in it. The tap is closed and A is filled with water to a depth of 4 units. The tap is then opened. When the water comes to rest, both surfaces are at a height h above the bottom of B, as shown in the diagram. Show that h satisfies the equation

$$h^2 - 3h + \sinh h = 15.$$

The initial volume of water in A is:

$$\pi \int_0^4 x^2 \,\mathrm{d}y = \pi \int_0^4 y \mathrm{d}y$$

$$= \pi[\frac{y^2}{2}]_0^4$$
$$= 8\pi$$

We assume that no water is in the tube as it is 'thin'.
Therefore we must have:

$$\begin{aligned}
8\pi &= \pi \int_0^{h-1} x^2 \mathrm{d}y + \pi \int_0^h x^2 \mathrm{d}y \\
&= \pi \int_0^{h-1} y \mathrm{d}y + \pi \int_0^h \left(\sinh \frac{x}{2}\right)^2 \mathrm{d}y \\
&= \pi \left[\frac{y^2}{2}\right]_0^{h-1} + \pi \int_0^h \frac{-1+\cosh y}{2} \mathrm{d}y \\
&= \pi \frac{(h-1)^2}{2} + \pi \left[-\frac{y}{2} + \frac{\sinh y}{2}\right]_0^h \\
&= \pi \frac{(h-1)^2}{2} - \pi \frac{h}{2} + \pi \frac{\sinh h}{2} \\
\Rightarrow \quad 0 &= h^2 - 2h + 1 - h + \sinh h - 16 \\
&= h^2 - 3h + \sinh h - 15 \\
\Rightarrow \quad 15 &= h^2 - 3h + \sinh h
\end{aligned}$$

Question (1992 STEP II Q8)

Calculate the following integrals

(i) $\displaystyle\int \frac{x}{(x-1)(x^2-1)} \,\mathrm{d}x;$

(ii) $\displaystyle\int \frac{1}{3\cos x + 4\sin x} \,\mathrm{d}x;$

(iii) $\displaystyle\int \frac{1}{\sinh x} \,\mathrm{d}x.$

(i)

$$\begin{aligned}
\int \frac{x}{(x-1)(x^2-1)} \mathrm{d}x &= \int \frac{x}{(x-1)^2(x+1)} \mathrm{d}x \\
&= \int \frac{1}{2(x-1)^2} + \frac{1}{4(x-1)} - \frac{1}{4(x+1)} \mathrm{d}x \\
&= -\frac{1}{2}(x-1)^{-1} + \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) + C
\end{aligned}$$

(ii)

$$\int \frac{1}{3\cos x + 4\sin x} \mathrm{d}x = \int \frac{1}{5\cos(x - \cos^{-1}(3/5))} \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{5}\int \sec(x - \cos^{-1}(3/5))\mathrm{d}x \\
&= \frac{1}{5}\left(\ln|\sec(x - \cos^{-1}(3/5)) + \tan(x - \cos^{-1}(3/5))|\right) + C
\end{aligned}
$$

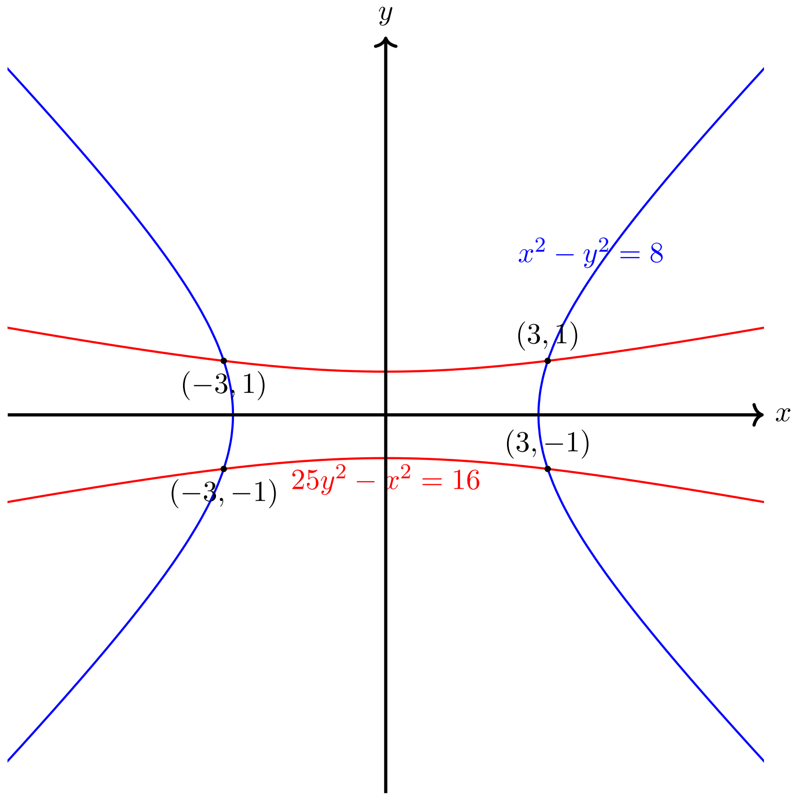
(iii)

$$
\begin{aligned}
\int \frac{1}{\sinh x}\mathrm{d}x &= \int \frac{2}{e^x - e^{-x}} \\
&= \int \frac{2e^x}{e^{2x} - 1}\mathrm{d}x \\
&= \int \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}\mathrm{d}x \\
&= \ln(e^x - 1) + \ln(e^x + 1) + C
\end{aligned}
$$

Question (2001 STEP III Q2)

Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$. Show that the area of the region defined by the inequalities $y^2 \geq x^2 - 8$ and $x^2 \geq 25y^2 - 16$ is $(72/5)\ln 2$.

$$
\begin{aligned}
& & x &= \cosh y \\
\Rightarrow & & x &= \tfrac{1}{2}(e^y + e^{-y}) \\
\Rightarrow & & 0 &= e^{2y} - 2xe^y + 1 \\
\Rightarrow & & e^y &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\
& & &= x \pm \sqrt{x^2 - 1} \\
\Rightarrow & & e^y &= x + \sqrt{x^2 - 1} & & \text{(by convention } \cosh^{-1} > 0) \\
\Rightarrow & & y &= \ln(x + \sqrt{x^2 - 1})
\end{aligned}
$$



$$A = 4\left(\int_0^3 \frac{1}{5}\sqrt{16+x^2}\mathrm{d}x - \int_{2\sqrt{2}}^3 \sqrt{x^2-8}\mathrm{d}x\right)$$

$$x = 4\sinh u: \qquad \int_0^3 \sqrt{4^2+x^2}\mathrm{d}x = \int_{u=0}^{u=\sinh^{-1}(3/4)} \sqrt{4^2(1+\sinh^2 u)}4\cosh u\mathrm{d}u$$

$$= \int_0^{\sinh^{-1}(3/4)} 16\cosh^2 u\mathrm{d}u$$

$$= 8\int_0^{\sinh^{-1}(3/4)} (1+\cosh 2u)\mathrm{d}u$$

$$= 8\left[u + \frac{1}{2}\sinh 2u\right]_0^{\sinh^{-1}(3/4)}$$

$$= 8\left(\sinh^{-1}(3/4) + \frac{1}{2}\sinh\left(2\sinh^{-1}(3/4)\right)\right)$$

$$\sinh^{-1}(3/4) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^2 + 1}\right)$$

$$= \ln\left(\frac{3}{4} + \frac{5}{4}\right)$$

$$= \ln 2$$

$$\Rightarrow \qquad \int_0^3 \sqrt{4^2+x^2}\mathrm{d}x = 8\ln 2 + 4\left(\frac{e^{2\ln 2} - e^{-2\ln 2}}{2}\right)$$

$$= 8\ln 2 + 2\cdot 4 - 2\cdot\frac{1}{4}$$
$$= 8\ln 2 + \frac{15}{2}$$

$$x = 2\sqrt{2}\cosh u: \quad \int_{2\sqrt{2}}^{3}\sqrt{x^2-8}\,\mathrm{d}x = \int_{u=0}^{u=\cosh^{-1}\frac{3}{2\sqrt{2}}}\sqrt{8(\cosh^2 u - 1)}2\sqrt{2}\sinh u\mathrm{d}u$$
$$= \int_0^{\cosh^{-1}\frac{3}{2\sqrt{2}}} 8\sinh^2 u\mathrm{d}u$$
$$= 4\int_0^{\cosh^{-1}\frac{3}{2\sqrt{2}}} 2\sinh^2 u\mathrm{d}u$$
$$= 4\int_0^{\cosh^{-1}\frac{3}{2\sqrt{2}}} \cosh 2u - 1\mathrm{d}u$$
$$= 4\left[\frac{1}{2}\sinh 2u - u\right]_0^{\cosh^{-1}\frac{3}{2\sqrt{2}}}$$

$$\cosh^{-1}\frac{3}{2\sqrt{2}} = \ln\left(\frac{3}{2\sqrt{2}} + \sqrt{\left(\frac{3}{2\sqrt{2}}\right)^2 - 1}\right)$$
$$= \ln\left(\frac{3}{2\sqrt{2}} + \sqrt{\frac{9}{8} - 1}\right)$$
$$= \ln\left(\frac{3}{2\sqrt{2}} + \sqrt{\frac{1}{8}}\right)$$
$$= \ln\frac{4}{2\sqrt{2}}$$
$$= \frac{1}{2}\ln 2$$

$$\int_{2\sqrt{2}}^{3}\sqrt{x^2-8}\,\mathrm{d}x = 4\left(\frac{1}{2}\frac{e^{\ln 2} - e^{-\ln 2}}{2} - \frac{1}{2}\ln 2\right)$$
$$= 2 - \frac{1}{2} - 2\ln 2$$
$$= \frac{3}{2} - 2\ln 2$$

$$A = 4\left(\frac{1}{5}\left(8\ln 2 + \frac{15}{2}\right) - \left(\frac{3}{2} - 2\ln 2\right)\right)$$
$$= 4\cdot\left(\frac{8}{5} + 2\right)\ln 2$$
$$= \frac{72}{5}\ln 2$$

Question (2003 STEP III Q1)

Given that $x+a>0$ and $x+b>0$, and that $b>a$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right)=\frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

and find $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+b}{x+a}\right)$.

Hence, or otherwise, integrate, for $x>-1$,

(i)

$$\int\frac{1}{(x+1)\sqrt{x+3}}\mathrm{d}x$$

(ii)

$$\int\frac{1}{(x+3)\sqrt{x+1}}\mathrm{d}x$$

.

[You may use the results $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x=\frac{1}{\sqrt{1-x^2}}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\,arcosh\,x=\frac{1}{\sqrt{x^2-1}}$.]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right)&=\frac{1}{\sqrt{1-\left(\frac{x+a}{x+b}\right)^2}}\left(\frac{b-a}{(x+b)^2}\right)\\&=\frac{b-a}{(x+b)\sqrt{(x+b)^2-(x+a)^2}}\\&=\frac{b-a}{(x+b)\sqrt{(b-a)(2x+b+a)}}\\&=\frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+b}{x+a}\right)&=\frac{1}{\sqrt{\left(\frac{x+b}{x+a}\right)^2-1}}\left(-\frac{b-a}{(x+a)^2}\right)\\&=-\frac{b-a}{(x+a)\sqrt{(x+b)^2-(x+a)^2}}\\&=-\frac{b-a}{(x+a)\sqrt{(b-a)(a+b+2x)}}\\&=-\frac{\sqrt{b-a}}{(x+a)\sqrt{a+b+2x}}\end{aligned}$$

(i)

$$\begin{aligned}\int\frac{1}{(x+1)\sqrt{x+3}}\mathrm{d}x&=\int\frac{1}{(x+1)\sqrt{\frac{1}{2}(2x+6)}}\mathrm{d}x\\&=\int\frac{\sqrt{2}}{(x+1)\sqrt{2x+1+5}}\mathrm{d}x\end{aligned}$$

$$= \frac{\sqrt{2}}{2} \int \frac{\sqrt{5-1}}{(x+1)\sqrt{2x+1+5}} \mathrm{d}x$$
$$= -\frac{\sqrt{2}}{2} \operatorname{arcosh}\left(\frac{x+5}{x+1}\right) + C$$

(ii)

$$\int \frac{1}{(x+3)\sqrt{x+1}} \mathrm{d}x = \int \frac{1}{(x+3)\sqrt{\frac{1}{2}(2x+2)}} \mathrm{d}x + C$$
$$= \int \frac{\sqrt{3-1}}{(x+3)\sqrt{2x+3-1}} \mathrm{d}x$$
$$= \arcsin\left(\frac{x-1}{x+3}\right)$$

Question (2004 STEP III Q1)

Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1} \,\mathrm{d}x = \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x} \,\mathrm{d}x \,.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x} \,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) .$$

By substituting $u = \mathrm{e}^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4} \,\mathrm{d}u \,.$$

Question (2005 STEP III Q7)

Show that if $\int \frac{1}{u\,f(u)} \,\mathrm{d}u = (u) + c$, then $\int \frac{m}{x\,f(x^m)} \,\mathrm{d}x = (x^m) + c$, where $m \neq 0$.

Find:

(i) $\int \frac{1}{x^n - x} \,\mathrm{d}x$;

(ii) $\int \frac{1}{\sqrt{x^n + x^2}} \,\mathrm{d}x$.

$u = x^m, \mathrm{d}u = mx^{m-1}$

$$\int \frac{m}{xf(x^m)} \mathrm{d}x = \int \frac{mx^{m-1}}{uf(u)} \mathrm{d}x$$
$$= \int \frac{1}{uf(u)} \mathrm{d}u$$

$$= F(u) + c$$
$$= F(x^m) + c$$

(i)

$$\int \frac{1}{u(u-1)} \mathrm{d}u = \int \left(\frac{1}{u-1} - \frac{1}{u} \right) \mathrm{d}u$$
$$= \ln \left(\frac{u-1}{u} \right) + c$$
$$= \ln \left(1 - \frac{1}{u} \right) + c$$
$$\int \frac{1}{x^n - x} \mathrm{d}x = \int \frac{1}{x(x^{n-1}-1)} \mathrm{d}x$$
$$f(u) = u - 1: \qquad = \frac{1}{n-1} \ln \left(1 - \frac{1}{x^{n-1}} \right) + c$$

(ii)

$$v = \sqrt{u+1}, \mathrm{d}v = \tfrac{1}{2}(u+1)^{-1/2}\mathrm{d}u \qquad \int \frac{1}{u\sqrt{u+1}} \mathrm{d}u = \int \frac{1}{(v^2-1)}(u+1)^{-1/2}\mathrm{d}u$$
$$= \int \frac{2}{v^2-1} \mathrm{d}v$$
$$= \ln \frac{1-v}{1+v} + c$$
$$= \ln \left(\frac{1-\sqrt{u+1}}{1+\sqrt{u+1}} \right) + c$$
$$f(u) = \sqrt{x+1}: \quad \int \frac{1}{\sqrt{x^n + x^2}} \mathrm{d}x = \int \frac{1}{x\sqrt{x^{n-2}+1}} \mathrm{d}x$$
$$= \frac{1}{n-2} \ln \left(\frac{1-\sqrt{x^{n-2}+1}}{1+\sqrt{x^{n-2}+1}} \right) + c$$

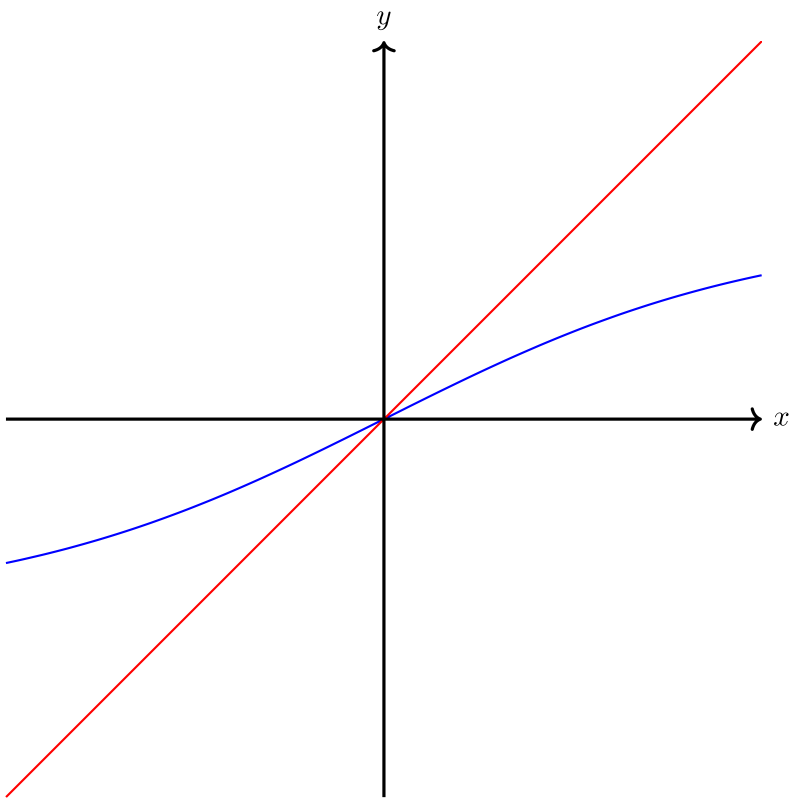
Question (2008 STEP III Q4) **(i)** Show, with the aid of a sketch, that $y > \tanh(y/2)$ for $y > 0$ and deduce that

$$\text{arcosh}x > \frac{x-1}{\sqrt{x^2-1}} \text{ for } x > 1. \qquad (*)$$

(ii) By integrating $(*)$, show that $\text{arcosh}x > 2\dfrac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.

(iii) Show that $\text{arcosh}x > 3\dfrac{\sqrt{x^2-1}}{x+2}$ for $x > 1$.

[**Note:** $\text{arcosh}x$ is another notation for $\cosh^{-1} x$.]



(i)

If $y = \text{arcosh}x$, then $\tanh \text{arcosh}x/2 = \sqrt{\frac{\cosh \text{arcosh}x - 1}{\cosh \text{arcosh}x + 1}} = \sqrt{\frac{x-1}{x+1}} = \frac{x-1}{\sqrt{x^2-1}}$

(ii)

$$\begin{aligned}\int \text{arcosh}x\text{d}x &= [x\text{arcosh}x] - \int \frac{x}{\sqrt{x^2-1}}\text{d}x \\ &= x\text{arcosh}x - \sqrt{x^2-1} + C \\ \int \frac{x-1}{\sqrt{x^2-1}} &= \sqrt{x^2-1} - \text{arcosh}x + C\end{aligned}$$

Therefore

$$\begin{aligned} & \int_1^x \text{arcosh}t\text{d}t > \int_1^x \frac{t-1}{\sqrt{t^2-1}}\text{d}t \\ \Rightarrow \quad & x\text{arcosh}x - \sqrt{x^2-1} - 0 > \sqrt{x^2-1} - \text{arcosh}x - 0 \\ \Rightarrow \quad & (x+1)\text{arcosh}x > 2\sqrt{x^2-1} \\ \Rightarrow \quad & \text{arcosh}x > 2\frac{\sqrt{x^2-1}}{x+1} \\ & = 2\frac{\sqrt{x-1}}{\sqrt{x+1}} \\ & = 2\frac{x-1}{\sqrt{x^2-1}}\end{aligned}$$

(iii) Integrating both sides again,

$$\begin{aligned} & \int_1^x \text{arcosh} t \text{d}t > 2\int_1^x \frac{t-1}{\sqrt{t^2-1}}\text{d}t \\ \Rightarrow \quad & x\text{arcosh}x - \sqrt{x^2-1} > 2\left(\sqrt{x^2-1} - \text{arcosh}x\right) \\ \Rightarrow \quad & (x+2)\text{arcosh}x > 3\sqrt{x^2-1} \\ \Rightarrow \quad & \text{arcosh}x > 3\frac{\sqrt{x^2-1}}{x+2} \end{aligned}$$

Question (2010 STEP III Q2)

In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials. By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2+2x\cosh a + 1}\,\text{d}x = \frac{a}{2\sinh a}\,.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2+2x\sinh a - 1}\,\text{d}x$$

and

$$\int_0^\infty \frac{1}{x^4+2x^2\cosh a + 1}\,\text{d}x\,.$$

(i) $\cosh a = \frac{1}{2}(e^a + e^{-a})$

$$\begin{aligned} \int_0^1 \frac{1}{x^2+2x\cosh a+1}\,\text{d}x &= \int_0^1 \frac{1}{x^2+(e^a+e^{-a})x+e^ae^{-a}}\text{d}x \\ &= \int_0^1 \frac{1}{e^a-e^{-a}}\left(\frac{1}{x+e^{-a}} - \frac{1}{x+e^a}\right)\text{d}x \\ &= \frac{1}{2\sinh a}\int_0^1 \left(\frac{1}{x+e^{-a}} - \frac{1}{x+e^a}\right)\text{d}x \\ &= \frac{1}{2\sinh a}\left[\ln(x+e^{-a}) - \ln(x+e^a)\right]_0^1 \\ &= \frac{1}{2\sinh a}\left(\ln(1+e^a) - \ln(1+e^{-a}) - (\ln e^{-a} - \ln e^a)\right) \\ &= \frac{1}{2\sinh a}\left(2a + \ln\frac{1+e^a}{1+e^{-a}}\right) \\ &= \frac{1}{2\sinh a}(2a-a) \\ &= \frac{a}{2\sinh a} \end{aligned}$$

(ii)

$$\int_1^\infty \frac{1}{x^2+2x\sinh a-1}\,\text{d}x = \int_1^\infty \frac{1}{(x+e^a)(x-e^{-a})}\text{d}x$$

$$
\begin{aligned}
&= \int_1^\infty \frac{1}{e^a + e^{-a}} \left(\frac{1}{x - e^{-a}} - \frac{1}{x + e^a} \right) \mathrm{d}x \\
&= \frac{1}{2\cosh a} \int_1^\infty \left(\frac{1}{x - e^{-a}} - \frac{1}{x + e^a} \right) \mathrm{d}x \\
&= \frac{1}{2\cosh a} \left[\ln(x - e^{-a}) - \ln(x + e^a) \right]_1^\infty \\
&= \frac{1}{2\cosh a} \left[\ln \frac{x - e^{-a}}{x + e^a} \right]_1^\infty \\
&= \frac{1}{2\cosh a} \left(0 - \ln \frac{1 - e^{-a}}{1 + e^a} \right) \\
&= \frac{1}{2\cosh a} \ln \frac{1 + e^a}{1 - e^{-a}} \\
&= \frac{1}{2\cosh a} \left(a + \ln \coth \frac{a}{2} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\infty \frac{1}{x^4 + 2x^2 \cosh a + 1} \, \mathrm{d}x &= \int_0^\infty \frac{1}{(x^2 + e^a)(x^2 + e^{-a})} \mathrm{d}x \\
&= \int_0^\infty \frac{1}{e^a - e^{-a}} \left(\frac{1}{x^2 + e^{-a}} - \frac{1}{x^2 + e^a} \right) \mathrm{d}x \\
&= \frac{1}{2\sinh a} \left[\frac{1}{e^{-a/2}} \tan^{-1} \frac{x}{e^{-a/2}} - \frac{1}{e^{a/2}} \tan^{-1} \frac{x}{e^{a/2}} \right]_0^\infty \\
&= \frac{1}{2\sinh a} \left(e^{a/2} \frac{\pi}{2} - e^{-a/2} \frac{\pi}{2} - 0 \right) \\
&= \frac{1}{2\sinh a} \pi \sinh \frac{a}{2} \\
&= \frac{\pi \sinh \frac{a}{2}}{2\sinh a} \\
&= \frac{\pi \sinh \frac{a}{2}}{4 \sinh \frac{a}{2} \cosh \frac{a}{2}} \\
&= \frac{\pi}{4 \cosh \frac{a}{2}}
\end{aligned}
$$

Question (2011 STEP III Q4)

The following result applies to any function f which is continuous, has positive gradient and satisfies $f(0)=0$:

$$ab \leq \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y\,, \qquad (*)$$

where f^{-1} denotes the inverse function of f, and $a \geq 0$ and $b \geq 0$.

(i) By considering the graph of $y = f(x)$, explain briefly why the inequality $(*)$ holds. In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

(ii) By taking $f(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leq a \leq \frac{1}{2}\pi$ and $0 \leq b \leq 1$,

$$ab \leq b\arcsin b + \sqrt{1-b^2} - \cos a\,.$$

Deduce that, for $t \geq 1$,

$$\arcsin(t^{-1}) \geq t - \sqrt{t^2-1}\,.$$

Question (2011 STEP III Q6)

The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t\,, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u\,,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2}\,\mathrm{d}v\,, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,\mathrm{d}x\,.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

$$\begin{aligned}
T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t \\
&= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{2t} \ln\left(\frac{1+t}{1-t}\right)\,\mathrm{d}t \\
u = \tfrac{1+t}{1-t}, t = \tfrac{u-1}{u+1}, \mathrm{d}t = \tfrac{2}{(u+1)^2}\mathrm{d}t \qquad &= \int_{u=2}^{u=3} \frac{1}{2t} \ln u \frac{2}{(u+1)^2}\mathrm{d}u \\
&= \int_2^3 \frac{u+1}{u-1} \ln u \frac{1}{(u+1)^2}\mathrm{d}u
\end{aligned}$$

$$= \int_2^3 \frac{1}{u^2-1} \ln u \mathrm{d}u$$

$$U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u} \mathrm{d}u$$

$v = e^u, \mathrm{d}v = e^u \mathrm{d}u$

$$= \int_{v=2}^{v=3} \frac{\ln v}{v - \frac{1}{v}} \frac{1}{v} \mathrm{d}v$$

$$= \int_2^3 \frac{1}{v^2-1} \ln v \mathrm{d}v$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2} \mathrm{d}v$$

$u = \frac{1}{v}, \mathrm{d}u = -\frac{1}{v^2}\mathrm{d}v$

$$= -\int_{u=3}^{u=2} \frac{-\ln u}{1-\frac{1}{u^2}} \frac{-1}{u^2} \mathrm{d}u$$

$$= -\int_3^2 \frac{\ln u}{u^2-1} \mathrm{d}u$$

$$= \int_2^3 \frac{1}{u^2-1} \ln u \mathrm{d}u$$

$$X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x) \, \mathrm{d}x$$

$u = \coth x, \mathrm{d}u = (1-u^2)\mathrm{d}x$

$$= \int_{u=3}^{u=2} \ln u \frac{1}{1-u^2} \mathrm{d}u$$

$$= \int_2^3 \frac{\ln u}{u^2-1} \mathrm{d}u$$

Therefore all integrals are equal to the same integral, namely $\displaystyle\int_2^3 \frac{\ln u}{u^2-1} \mathrm{d}u$

Question (2014 STEP III Q2) **(i)** Show, by means of the substitution $u = \cosh x$, that

$$\int \frac{\sinh x}{\cosh 2x} \mathrm{d}x = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| + C \,.$$

(ii) Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x} \mathrm{d}x \,.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1+u^4} \mathrm{d}u = \frac{\pi + 2\ln(\sqrt{2}+1)}{4\sqrt{2}} \,.$$

(i)

$$\int \frac{\sinh x}{\cosh 2x}\mathrm{d}x = \int \frac{\sinh x}{2\cosh^2 x - 1}\mathrm{d}x$$

$$u = \cosh x, \mathrm{d}u = \sinh x \mathrm{d}x \qquad = \int \frac{1}{2u^2 - 1}\mathrm{d}u$$

$$= \int \frac{1}{2}\left(\frac{1}{\sqrt{2}u - 1} - \frac{1}{\sqrt{2}u + 1}\right)\mathrm{d}u$$

$$= \frac{1}{2\sqrt{2}}\left(\ln(\sqrt{2}u - 1) - \ln(\sqrt{2}u + 1)\right) + C$$

$$= \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1}\right) + C$$

(ii)

$$\int \frac{\cosh x}{\cosh 2x}\mathrm{d}x = \int \frac{\cosh x}{1 + 2\sinh^2 x}\mathrm{d}x$$

$$u = \sinh x \qquad = \int \frac{1}{1 + 2u^2}\mathrm{d}u$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}u) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\sinh x) + C$$

(iii)

$$u = e^x: \quad \int_0^1 \frac{1}{1 + u^4}\mathrm{d}u = \int_{x=-\infty}^{x=0} \frac{1}{1 + e^{4x}}e^x\mathrm{d}x$$

$$= \int_{-\infty}^{0} \frac{e^{-x}}{e^{2x} + e^{-2x}}\mathrm{d}x$$

$$= \int_{-\infty}^{0} \frac{\cosh x - \sinh x}{2\cosh 2x}\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{\cosh x}{\cosh 2x}\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{0} \frac{\sinh x}{\cosh 2x}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\sinh x)\right]_{-\infty}^{0} - \frac{1}{2}\left[\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1}\right)\right]_{-\infty}^{0}$$

$$= 0 - \frac{1}{2\sqrt{2}}\frac{-\pi}{2} - \left(\frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right) - 0\right)$$

$$= \frac{\pi - \ln((\sqrt{2} - 1)^2)}{4\sqrt{2}}$$

$$= \frac{\pi + 2\ln(1 + \sqrt{2})}{4\sqrt{2}}$$

Question (2025 STEP III Q7)

Let $f(x) = \sqrt{x^2+1} - x$.

(i) Using a binomial series, or otherwise, show that, for large $|x|$, $\sqrt{x^2+1} \approx |x| + \frac{1}{2|x|}$. Sketch the graph $y = f(x)$.

(ii) Let $g(x) = \tan^{-1} f(x)$ and, for $x \neq 0$, let $k(x) = \frac{1}{2}\tan^{-1}\frac{1}{x}$.

a) Show that $g(x) + g(-x) = \frac{1}{2}\pi$.

b) Show that $k(x) + k(-x) = 0$.

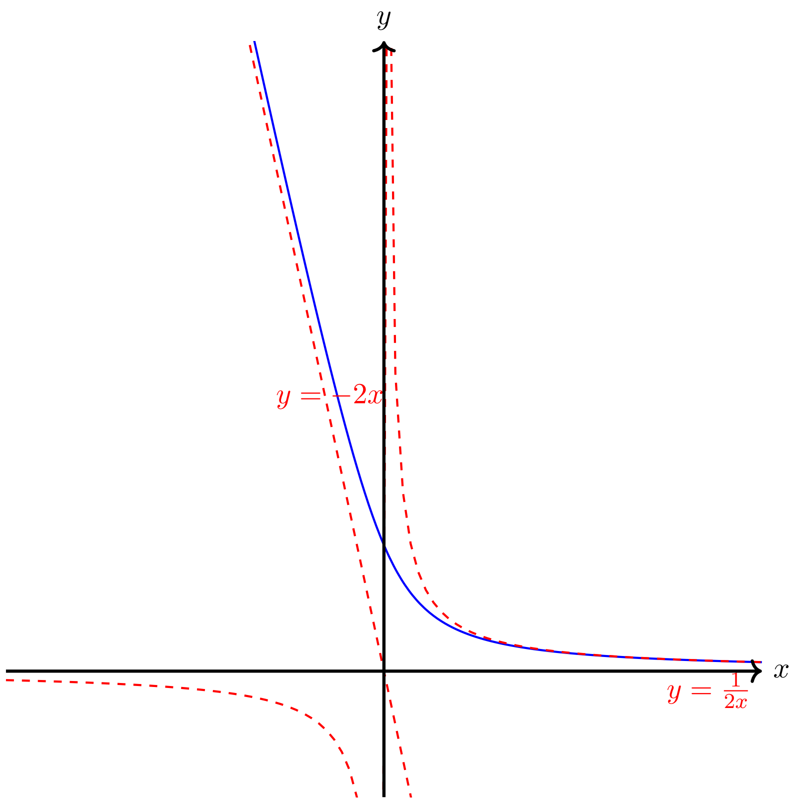
c) Show that $\tan k(x) = \tan g(x)$ for $x > 0$.

d) Sketch the graphs $y = g(x)$ and $y = k(x)$ on the same axes.

e) Evaluate $\int_0^1 k(x)\,dx$ and hence write down the value of $\int_{-1}^0 g(x)\,dx$.

(i)

$$\begin{aligned}
\sqrt{x^2+1} &= |x|\sqrt{1+\frac{1}{x^2}} \\
&= |x|\left(1+\frac{1}{2}\frac{1}{x^2}+\cdots\right) \qquad \text{if } \left(\frac{1}{x^2}<1\right) \\
&= |x| + \frac{1}{2}\frac{1}{|x|}+\cdots \\
&\approx |x| + \frac{1}{2|x|}
\end{aligned}$$



(ii) a)

$$\begin{aligned}\tan(g(x)+g(-x)) &= \tan\left(\tan^{-1}(\sqrt{x^2+1}-x)+\tan^{-1}(\sqrt{x^2+1}+x)\right)\\ &= \frac{\sqrt{x^2+1}-x+\sqrt{x^2+1}+x}{1-1}\\ \Rightarrow \quad g(x)+g(-x) &\in \left\{\cdots,-\frac{\pi}{2},\frac{\pi}{2},\cdots\right\}\end{aligned}$$

But $g(x), g(-x) > 0$ and $g(x), g(-x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, therefore it must be $\frac{\pi}{2}$.

b)

$$\begin{aligned}\tan(2(k(x)+k(-x))) &= \tan(\tan^{-1}x+\tan^{-1}(-x))\\ &= 0\\ \Rightarrow \quad k(x)+k(-x) &\in \left\{\cdots,-\frac{\pi}{2},0,\frac{\pi}{2},\cdots\right\}\end{aligned}$$

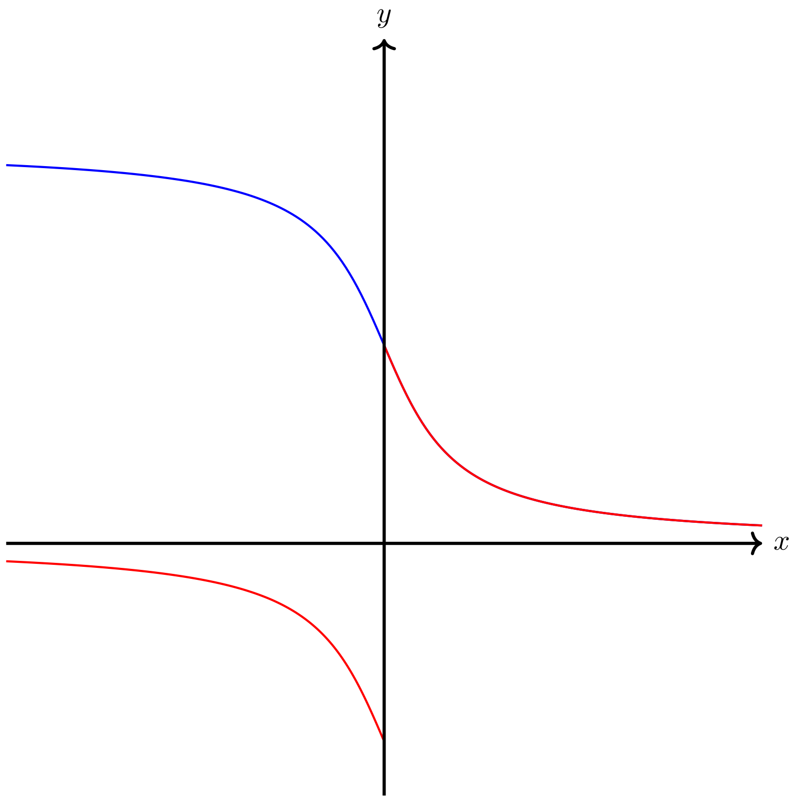
But $k(x) \in (-\frac{\pi}{4}, \frac{\pi}{4})$, therefore $k(x)+k(-x)=0$.

c) Let $t = \tan k(x)$.

$$\begin{aligned}\tan\left(\tan^{-1}\frac{1}{x}\right) &= \frac{2\tan\left(\frac{1}{2}\tan^{-1}\frac{1}{x}\right)}{1-\tan^2\left(\frac{1}{2}\tan^{-1}\frac{1}{x}\right)}\\ \Rightarrow \quad \frac{1}{x} &= \frac{2t}{1-t^2}\\ \Rightarrow \quad 1-t^2 &= 2tx\end{aligned}$$

$$\Rightarrow \quad 0 = t^2 + 2tx - 1$$
$$\Rightarrow \quad 0 = (t+x)^2 - 1 - x^2$$
$$\Rightarrow \quad t = -x \pm \sqrt{1+x^2}$$

Since $t > 0$, $t = \sqrt{1+x^2} - x = f(x) = \tan g(x)$



d)

e)

$$\begin{aligned}\int_0^1 k(x)\mathrm{d}x &= \int_0^1 \frac{1}{2}\tan^{-1}\left(\frac{1}{x}\right)\mathrm{d}x \\ &= \left[\frac{x}{2}\tan^{-1}\left(\frac{1}{x}\right)\right]_0^1 - \int_0^1 \frac{x}{2}\frac{-1/x^2}{1+1/x^2}\mathrm{d}x \\ &= \left[\frac{x}{2}\tan^{-1}\left(\frac{1}{x}\right)\right]_0^1 + \frac{1}{4}\int_0^1 \frac{2x}{1+x^2}\mathrm{d}x \\ &= \frac{1}{2}\frac{\pi}{4} + \frac{1}{4}\ln(2) \\ &= \frac{\pi + \ln 4}{8}\end{aligned}$$

Therefore $\int_{-1}^0 g(x)\mathrm{d}x = -\frac{\pi + \ln 4}{8}$

Question (1988 STEP II Q10)

The surface S in 3-dimensional space is described by the equation

$$\mathbf{a}\cdot\mathbf{r} + ar = a^2,$$

where $\mathbf{r}$ is the position vector with respect to the origin O, $\mathbf{a}(\neq \mathbf{0})$ is the position vector of a fixed point, $r = |\mathbf{r}|$ and $a = |\mathbf{a}|$. Show, with the aid of a diagram, that S is the locus of points which are equidistant from the origin O and the plane $\mathbf{r}\cdot\mathbf{a} = a^2$. The point P, with position vector $\mathbf{p}$, lies in S, and the line joining P to O meets S again at Q. Find the position vector of Q. The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ meets S at T and T'. Show that the position vectors of T and T' are

$$\pm\frac{1}{\sqrt{2ap - a^2}}\mathbf{a}\times\mathbf{p},$$

where $p = |\mathbf{p}|$. Show that the area of the triangle PQT is

$$\frac{ap^2}{2p - a}.$$

The plane is the same as the plane $(\mathbf{r}-\mathbf{a})\cdot\mathbf{a} = 0$, ie the plane through $\mathbf{a}$ whose normal is parallel to $\mathbf{a}$

The distance from $\mathbf{r}$ to the plane therefore is λ where $\mathbf{r} + \lambda\frac{1}{a}\mathbf{a}$ must be on the plane, ie $(\mathbf{r} + \frac{\lambda}{a}\mathbf{a} - \mathbf{a})\cdot\mathbf{a} = 0 \Rightarrow \lambda = \frac{a^2 - \mathbf{a}\cdot\mathbf{r}}{a}$

But if $\mathbf{a}\cdot\mathbf{r} = a^2 - ar$ then $\lambda = r$, ie the distance to the plane is the same as the distance to the origin.

$\mathbf{q} = k\mathbf{p}$ and so $\mathbf{a}\cdot k\mathbf{p} + a|k|p = a^2$ if $k > 0$ we will find $k = 1$ the position vector we already know about, therefore suppose $k < 0$ so:

$$\begin{aligned}
& & \mathbf{a}\cdot k\mathbf{p} - kap &= a^2 \\
&\Rightarrow & k(a^2 - ap) - kap &= a^2 \\
&\Rightarrow & k(a^2 - 2ap) &= a^2 \\
&\Rightarrow & k &= \frac{a^2}{a^2 - 2ap}
\end{aligned}$$

Therefore $\mathbf{q} = \frac{a^2}{a^2 - 2ap}\mathbf{p}$

The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ will be parallel to $\mathbf{a}\times\mathbf{p}$. Therefore we should consider points of the from $s\mathbf{a}\times\mathbf{p}$ on the surface S.

$$s\mathbf{a}\cdot(\mathbf{a}\times\mathbf{p}) + sa^2p|\sin\theta| = a^2$$

The angle between $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{p}}{ap} = \frac{a^2 - ap}{ap} \Rightarrow |\sin\theta| = \sqrt{1 - \frac{(a-p)^2}{p^2}} = \frac{1}{p}\sqrt{2ap - a^2}$

Therefore $sa^2\sqrt{2ap - a^2} = a^2 \Rightarrow s = \frac{1}{\sqrt{2ap - a^2}}$ and so the points are as required.

Noting that $|\mathbf{p}\times\mathbf{t}| = |\frac{1}{p\sin\theta}\mathbf{p}\times(\mathbf{p}\times\mathbf{a})| = |\frac{1}{p\sin\theta}p^2 a\sin\theta| = pa$

The area of triangle PQT is :

$$\frac{1}{2}|(\mathbf{p}-\mathbf{t})\times(\mathbf{q}-\mathbf{t})| = \frac{1}{2}|\mathbf{p}\times\mathbf{q} - \mathbf{t}\times\mathbf{q} - \mathbf{p}\times\mathbf{t} - \mathbf{t}\times\mathbf{t}|$$

$$\begin{aligned}
&= \frac{1}{2}|\mathbf{t} \times (\mathbf{p} - \mathbf{q})| \\
&= \frac{1}{2} \cdot (1 - \frac{a^2}{a^2 - 2ap})|\mathbf{t} \times \mathbf{p}| \\
&= \frac{1}{2} \frac{2ap}{a^2 - 2ap} \cdot ap \\
&= \frac{ap^2}{a^2 - ap}
\end{aligned}$$

Question (1989 STEP III Q2)

The points A, B and C lie on the surface of the ground, which is an inclined plane. The point B is 100m due north of A, and C is 60m due east of B. The vertical displacements from A to B, and from B to C, are each 5m downwards. A plane coal seam lies below the surface and is to be located by making vertical bore-holes at A, B and C. The bore-holes strike the coal seam at 95m, 45m and 76m below A, B and C respectively. Show that the coal seam is inclined at $\cos^{-1}(\frac{4}{5})$ to the horizontal. The coal seam comes to the surface along a line. Find the bearing of this line.

Set up a coordinate system so that x is E-W, y is N-S and z is the vertical direction. Also assume B is the origin, then, $A = \begin{pmatrix} 0 \\ -100 \\ 5 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 60 \\ 0 \\ -5 \end{pmatrix}$,.

The coal seam has points: $\begin{pmatrix} 0 \\ -100 \\ -90 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -45 \end{pmatrix}, \begin{pmatrix} 60 \\ 0 \\ -81 \end{pmatrix}$,

Therefore we can find the normal to the coal seam:

$$\begin{aligned}
\mathbf{n} &= \left(\begin{pmatrix} 0 \\ -100 \\ -90 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -45 \end{pmatrix} \right) \times \left(\begin{pmatrix} 60 \\ 0 \\ -81 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -45 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ -100 \\ -45 \end{pmatrix} \times \begin{pmatrix} 60 \\ 0 \\ -36 \end{pmatrix} \\
&= \begin{pmatrix} 3600 \\ -60 \cdot 45 \\ 60 \cdot 100 \end{pmatrix} \\
&= 300 \begin{pmatrix} 12 \\ -9 \\ 20 \end{pmatrix}
\end{aligned}$$

To measure the incline θ to the horizontal we can take a dot with $\hat{\mathbf{k}}$, to see:

$$\begin{aligned}
\cos\theta &= \frac{20}{\sqrt{12^2 + (-9)^2 + 20^2}\sqrt{1^2 + 0^2 + 0^2}} \\
&= \frac{20}{25} \\
&= \frac{4}{5}
\end{aligned}$$

Therefore the angle is $\cos^{-1}\frac{4}{5}$
The equation of the seam is $12x - 9y + 20z = -900$.
The equation of the surface is $5x + 3y + 60z = 0$
We can compute the direction of the overlap again with a cross product:

$$\begin{aligned}\mathbf{d} &= \begin{pmatrix} 12 \\ -9 \\ 20 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 60 \end{pmatrix} \\ &= \begin{pmatrix} -600 \\ -620 \\ 81 \end{pmatrix}\end{aligned}$$

To get the bearing of this vector we just need to look at the x and y components, so it will be $\tan^{-1}\frac{600}{620} = \tan^{-1}\frac{30}{31}$

Question (1992 STEP II Q9)

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane. Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r}\cdot\mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda + \mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \lambda\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) + \mu\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

for all scalars λ, μ. Deduce that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

Say briefly what happens if A, B, C and O are all in the same plane.

$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$

Therefore it is the plane through $\mathbf{a}$ with direction vectors $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$, ie it is the plane through $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

The normal to this plane will be $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}$, so we must have:

$$\begin{aligned}\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}) &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}) \\ &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{r}\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}) \\ &= ((1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}) \\ &= (1 - \lambda - \mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) - \lambda\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) - \mu\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}) \\ \Rightarrow \quad 0 &= (-\lambda - \mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) - \lambda\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) - \mu\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})\end{aligned}$$

$$= -(\lambda+\mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \lambda\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) + \mu\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

The result follows from setting $\mu = 0, \lambda = 1$ and $\mu = 1, \lambda = 0$.

If they all lie in the same plane then the plane described is through the origin, and those values are all the same, but equal to 0.

Question (1993 STEP II Q4)

Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\mathbf{m}_1\times\mathbf{m}_2)|}{|(\mathbf{m}_1\times\mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2,1,-1) \qquad \mathbf{p}_2 = (1,0,-2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4,3,0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0,-3,1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2\,|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2)\cdot(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

Question (1995 STEP III Q8)

A plane π in 3-dimensional space is given by the vector equation $\mathbf{r}\cdot\mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π. The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r}\cdot\mathbf{n}_i = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{n}_1\times\mathbf{n}_2) = 0$$

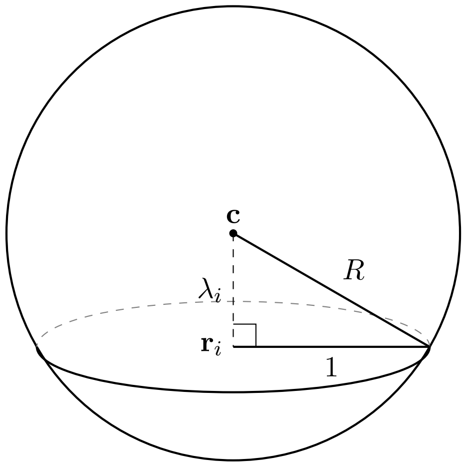
and that

$$(p_1 - \mathbf{n}_1\cdot\mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2\cdot\mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

The equation of the normal to π through X is $\mathbf{x} + \lambda\mathbf{n}$. The distance is $|\mathbf{x} \cdot \mathbf{n} - p|$

We know that the centre of the sphere must lie above the centre of the circle following the normal, ie $\mathbf{c} = \mathbf{r}_1 + \lambda_1\mathbf{n}_1 = \mathbf{r}_2 + \lambda_2\mathbf{n}_2$



We can also see that $R^2 = 1 + \lambda_1^2 = 1 + \lambda_2^2 \Rightarrow \lambda_1 = \pm\lambda_2$, from which we obtain the desired result.

Therefore the condition is

$$\begin{aligned}
\mathbf{r}_1 + \lambda\mathbf{n}_1 &= \mathbf{r}_2 \pm \lambda\mathbf{n}_2 \qquad (1)\\
\mathbf{r}_1 - \mathbf{r}_2 &= \lambda(\pm\mathbf{n}_1 - \mathbf{n}_2)\\
\Rightarrow \qquad (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) &= (\lambda(\pm\mathbf{n}_1 - \mathbf{n}_2)) \cdot (\mathbf{n}_1 \times \mathbf{n}_2)\\
&= \lambda\left(\pm\mathbf{n}_1 \cdot (\mathbf{n}_1 \times \mathbf{n}_2) - \mathbf{n}_2 \cdot (\mathbf{n}_1 \times \mathbf{n}_2)\right)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{n}_1 \cdot (1) \qquad \mathbf{r}_1 \cdot \mathbf{n}_1 + \lambda\mathbf{n}_1 \cdot \mathbf{n}_1 &= \mathbf{r}_2 \cdot \mathbf{n}_1 \pm \lambda\mathbf{n}_2 \cdot \mathbf{n}_1\\
p_1 + \lambda &= \mathbf{r}_2 \cdot \mathbf{n}_1 \pm \lambda\mathbf{n}_2 \cdot \mathbf{n}_1
\end{aligned}$$

$$\begin{aligned}
\mathbf{n}_2 \cdot (1) \qquad \mathbf{r}_1 \cdot \mathbf{n}_2 + \lambda\mathbf{n}_1 \cdot \mathbf{n}_2 &= \mathbf{r}_2 \cdot \mathbf{n}_2 \pm \lambda\mathbf{n}_2 \cdot \mathbf{n}_2\\
\mathbf{r}_1 \cdot \mathbf{n}_2 + \lambda\mathbf{n}_1 \cdot \mathbf{n}_2 &= p_2 \pm \lambda\\
\pm\lambda - \lambda\mathbf{n}_1 \cdot \mathbf{n}_2 &= \mathbf{r}_1 \cdot \mathbf{n}_2 - p_2\\
&= \pm(\mathbf{r}_2 \cdot \mathbf{n}_1 - p_1)\\
\Rightarrow \qquad (p_1 - \mathbf{n}_1 \cdot \mathbf{r}_2)^2 &= (p_2 - \mathbf{n}_2 \cdot \mathbf{r}_1)^2
\end{aligned}$$

The first condition means the line between the centres lies in the plane spanned by the normal of the two planes π_1 and π_2.

The second condition means that the distance of the center to the other plane is the same for both centres/planes.

Question (1998 STEP III Q8) **(i)** Show that the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$, where $\mathbf{m}$ is a unit vector, intersects the sphere $\mathbf{r}\cdot\mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}\cdot\mathbf{b} - (\mathbf{b}\cdot\mathbf{m})^2\,.$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\mathbf{m}\cdot\mathbf{p} = 0$.

(ii) Now consider a second sphere of radius a and a plane perpendicular to a unit vector $\mathbf{n}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is l. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (*i.e.* the two radii to each point of intersection are perpendicular) if

$$\mathbf{d}\cdot\mathbf{d} = 2a^2\,.$$

Question (2000 STEP II Q7)

The line l has vector equation $\mathbf{r} = \lambda\mathbf{s}$, where

$$\mathbf{s} = (\cos\theta + \sqrt{3}\,)\,\mathbf{i} + (\sqrt{2}\,\sin\theta)\,\mathbf{j} + (\cos\theta - \sqrt{3}\,)\,\mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between l and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between l and m is $\pi/6$.

A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

Question (2000 STEP III Q5)

Given two non-zero vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ define $\Delta(\mathbf{a},\mathbf{b})$ by $\Delta(\mathbf{a},\mathbf{b}) = a_1 b_2 - a_2 b_1$.

Let A, B and C be points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, no two of which are parallel. Let P, Q and R be points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, none of which are parallel.

(i) Show that there exists a 2×2 matrix $\mathbf{M}$ such that P and Q are the images of A and B under the transformation represented by $\mathbf{M}$.

(ii) Show that $\Delta(\mathbf{a},\mathbf{b})\,\mathbf{c} + \Delta(\mathbf{c},\mathbf{a})\,\mathbf{b} + \Delta(\mathbf{b},\mathbf{c})\,\mathbf{a} = 0$.

Hence, or otherwise, prove that a necessary and sufficient condition for the points P, Q, and R to be the images of points A, B and C under the transformation represented by some 2×2 matrix $\mathbf{M}$ is that

$$\Delta(\mathbf{a},\mathbf{b}) : \Delta(\mathbf{b},\mathbf{c}) : \Delta(\mathbf{c},\mathbf{a}) = \Delta(\mathbf{p},\mathbf{q}) : \Delta(\mathbf{q},\mathbf{r}) : \Delta(\mathbf{r},\mathbf{p})\,.$$

Question (2005 STEP II Q7)

The position vectors, relative to an origin O, at time t of the particles P and Q are

$$\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\,\mathbf{k} \text{ and } \cos(t + \tfrac{1}{4}\pi)\left[\tfrac{3}{2}\mathbf{i} + \tfrac{3\sqrt{3}}{2}\mathbf{k}\right] + 3\sin(t + \tfrac{1}{4}\pi)\,\mathbf{j}\,,$$

respectively, where $0 \le t \le 2\pi$.

(i) Give a geometrical description of the motion of P and Q.

(ii) Let θ be the angle POQ at time t that satisfies $0 \le \theta \le \pi$. Show that

$$\cos\theta = \tfrac{3\sqrt{2}}{8} - \tfrac{1}{4}\cos(2t + \tfrac{1}{4}\pi)\,.$$

(iii) Show that the total time for which $\theta \ge \frac{1}{4}\pi$ is $\frac{3}{2}\pi$.

(i) P is travelling in a unit circle about the origin in the $\mathbf{i} - \mathbf{j}$ plane. Q is travelling in a circle (also about the origin, but in a different plane with radius 3).

(ii)

$$\begin{aligned}
&& \mathbf{p}\cdot\mathbf{q} &= |\mathbf{p}||\mathbf{q}|\cos\theta \\
\Rightarrow && \cos\theta &= \frac{\frac{3}{2}\cos t\cos(t+\frac{\pi}{4}) + 3\sin t\sin(t+\frac{\pi}{4})}{3} \\
&& &= \tfrac{1}{2}\cos t\cos(t+\tfrac{\pi}{4}) + \sin t\sin(t+\tfrac{\pi}{4}) \\
&& &= \tfrac{1}{4}(\cos(2t+\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4})) + \tfrac{1}{2}(\cos(\tfrac{\pi}{4}) - \cos(2t+\tfrac{\pi}{4})) \\
&& &= \tfrac{3\sqrt{2}}{8} - \tfrac{1}{4}\cos(2t+\tfrac{\pi}{4})
\end{aligned}$$

(iii) If $\theta \ge \frac{1}{4}\pi$, then $\cos\theta \le \frac{\sqrt{2}}{2}$

$$\begin{aligned}
&& \frac{\sqrt{2}}{2} &\ge \frac{3\sqrt{2}}{8} - \frac{1}{4}\cos(2t+\tfrac{\pi}{4}) \\
\Rightarrow && \frac{\sqrt{2}}{2} &\ge -\cos(2t+\tfrac{\pi}{4}) \\
\Rightarrow && \cos(2t+\tfrac{\pi}{4}) &\ge -\frac{1}{\sqrt{2}} \\
\Rightarrow && 2t+\tfrac{\pi}{4} &\not\in (\tfrac{3\pi}{4}, \tfrac{5\pi}{4}) \cup (\tfrac{11\pi}{4}, \tfrac{13\pi}{4}) \\
\Rightarrow && t &\not\in (\tfrac{\pi}{4}, \tfrac{\pi}{2}) \cup (\tfrac{5\pi}{4}, \tfrac{3\pi}{2})
\end{aligned}$$

which is is a time of $\frac{\pi}{2}$, therefore the left over time is $\frac{3}{2}\pi$

Question (2006 STEP II Q8)
Show that the line through the points with position vectors $\mathbf{x}$ and $\mathbf{y}$ has equation

$$\mathbf{r} = (1 - \alpha)\mathbf{x} + \alpha\mathbf{y}\,,$$

where α is a scalar parameter. The sides OA and CB of a trapezium $OABC$ are parallel, and $OA > CB$. The point E on OA is such that $OE : EA = 1 : 2$, and F is the midpoint of CB. The point D is the intersection of OC produced and AB produced; the point G is the intersection of OB and EF; and the point H is the intersection of DG produced and OA. Let $\mathbf{a}$ and $\mathbf{c}$ be the position vectors of the points A and C, respectively, with respect to the origin O.

(i) Show that B has position vector $\lambda\mathbf{a} + \mathbf{c}$ for some scalar parameter λ.

(ii) Find, in terms of $\mathbf{a}$, $\mathbf{c}$ and λ only, the position vectors of D, E, F, G and H. Determine the ratio $OH : HA$.

Question (2007 STEP I Q7) **(i)** The line L_1 has vector equation $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-3\end{pmatrix}$.

The line L_2 has vector equation $\mathbf{r} = \begin{pmatrix}4\\-2\\9\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-2\end{pmatrix}$. Show that the distance D between a point on L_1 and a point on L_2 can be expressed in the form

$$D^2 = (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36\,.$$

Hence determine the minimum distance between these two lines and find the coordinates of the points on the two lines that are the minimum distance apart.

(ii) The line L_3 has vector equation $\mathbf{r} = \begin{pmatrix}2\\3\\5\end{pmatrix} + \alpha\begin{pmatrix}0\\1\\0\end{pmatrix}$. The line L_4 has vector equation $\mathbf{r} = \begin{pmatrix}3\\3\\-2\end{pmatrix} + \beta\begin{pmatrix}4k\\1-k\\-3k\end{pmatrix}$. Determine the minimum distance between these two lines, explaining geometrically the two different cases that arise according to the value of k.

Question (2007 STEP II Q8)

The points B and C have position vectors $\mathbf{b}$ and $\mathbf{c}$, respectively, relative to the origin A, and A, B and C are not collinear.

(i) The point X has position vector $s\mathbf{b}+t\mathbf{c}$. Describe the locus of X when $s+t=1$.

(ii) The point P has position vector $\beta\mathbf{b}+\gamma\mathbf{c}$, where β and γ are non-zero, and $\beta+\gamma\neq 1$. The line AP cuts the line BC at D. Show that $BD : DC = \gamma : \beta$.

(iii) The line BP cuts the line CA at E, and the line CP cuts the line AB at F. Show that

$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=1\,.$$

(i) X lies on the line including B and C.

(ii) points on the line AP have the form $\lambda(\beta\mathbf{b}+\gamma\mathbf{c})$, and the point D will be the point where $\lambda\beta+\lambda\gamma=1$.

$$\begin{aligned}\frac{|BD|}{|DC|} &= \frac{|\mathbf{b}-\lambda(\beta\mathbf{b}+\gamma\mathbf{c})|}{|\lambda(\beta\mathbf{b}+\gamma\mathbf{c})-\mathbf{c}|}\\ &= \frac{|(1-\lambda\beta)\mathbf{b}-\lambda\gamma\mathbf{c}|}{|\lambda\beta\mathbf{b}+(\lambda\gamma-1)\mathbf{c}|}\\ &= \frac{|\lambda\gamma\mathbf{b}-\lambda\gamma\mathbf{c}|}{|\lambda\beta\mathbf{b}-(\lambda\beta)\mathbf{c}|}\\ &= \frac{\gamma}{\beta}\end{aligned}$$

(iii) The line BP is $\mathbf{b}+\mu(\beta\mathbf{b}+\gamma\mathbf{c})$ and will meet CA when $1+\mu\beta=0$, ie $\mu=-\frac{1}{\beta}$, therefore E is $-\frac{\gamma}{\beta}\mathbf{c}$, and so $\frac{|CE|}{|EA|}=\frac{1+\gamma/\beta}{\gamma/\beta}=\frac{\beta+\gamma}{\gamma}$.

Similarly, F is $-\frac{\beta}{\gamma}\mathbf{b}$ and $\frac{|AF|}{|FB|}=\frac{\beta/\gamma}{1+\frac{\beta}{\gamma}}=\frac{\beta}{\gamma+\beta}$, and so

$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=\frac{\beta}{\gamma+\beta}\frac{\gamma}{\beta}\frac{\beta+\gamma}{\gamma}=1$$

Question (2008 STEP II Q8)

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points A, B and O are not collinear. The point P lies on AB between A and B such that

$$AP : PB = (1-\lambda) : \lambda\,.$$

Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. Given that OP bisects $\angle AOB$, determine λ in terms of a and b, where $a = |\mathbf{a}|$ and $b = ||$. The point Q also lies on AB between A and B, and is such that $AP = BQ$. Prove that

$$OQ^2 - OP^2 = (b-a)^2\,.$$

Question (2009 STEP II Q8)

The non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, given by

$$\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \quad \text{and} \quad \mathbf{q} = \mu\mathbf{a} + (1-\mu)\mathbf{c}$$

where $0 < \lambda < 1$ and $\mu > 1$. Draw a diagram showing A, B, C, P and Q. Given that $CQ \times BP = AB \times AC$, find μ in terms of λ, and show that, for all values of λ, the the line PQ passes through the fixed point D, with position vector $\mathbf{d}$ given by $\mathbf{d} = -a + b + c$. What can be said about the quadrilateral $ABDC$?

Question (2010 STEP I Q7)

Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta}\,,$$

lies on the lines OC and AB. The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

Question (2010 STEP II Q5)

The points A and B have position vectors $\mathbf{i}+j+k$ and $5\mathbf{i}-\mathbf{j}-\mathbf{k}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

(i) The line L_1 has equation $\mathbf{r}=\lambda(m\mathbf{i}+n\mathbf{j}+p\mathbf{k})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r}=\mu(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha=\angle AOB$, determine a relationship between u, v and w. Hence describe the surface with Cartesian equation $x^2+y^2+z^2=2(yz+zx+xy)$.

Question (2011 STEP II Q5)

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A. Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c}=\lambda\mathbf{a}-\mathbf{b}$$

where $\lambda=\dfrac{2\,\mathbf{a}.b}{\mathbf{a}.a}$. The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d}=\mu\mathbf{b}-\lambda\mathbf{a}$$

for some scalar μ to be determined. Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda=-\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

Question (2012 STEP II Q7)

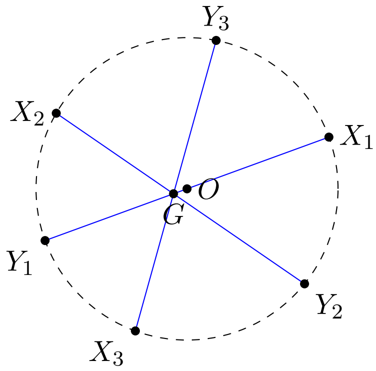
Three distinct points, X_1, X_2 and X_3, with position vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ respectively, lie on a circle of radius 1 with its centre at the origin O. The point G has position vector $\frac{1}{3}(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3)$. The line through X_1 and G meets the circle again at the point Y_1 and the points Y_2 and Y_3 are defined correspondingly. Given that $\overrightarrow{GY_1}=-\lambda_1\overrightarrow{GX_1}$, where λ_1 is a positive scalar, show that

$$\overrightarrow{OY_1}=\tfrac{1}{3}\big((1-2\lambda_1)\mathbf{x}_1+(1+\lambda_1)(\mathbf{x}_2+\mathbf{x}_3)\big)$$

and hence that

$$\lambda_1=\frac{3-\alpha-\beta-\gamma}{3+\alpha-2\beta-2\gamma}\,,$$

where $\alpha=\mathbf{x}_2\,.\,\mathbf{x}_3$, $\beta=\mathbf{x}_3\,.\,\mathbf{x}_1$ and $\gamma=\mathbf{x}_1\,.\,\mathbf{x}_2$. Deduce that $\dfrac{GX_1}{GY_1}+\dfrac{GX_2}{GY_2}+\dfrac{GX_3}{GY_3}=3\,.$



$$
\begin{aligned}
\mathbf{y}_1 &= \overrightarrow{OG} + \overrightarrow{GY_1} \\
&= \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \lambda_1 \left(\mathbf{x}_1 - \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)\right) \\
&= \frac{1}{3}\left((1-2\lambda_1)\mathbf{x}_1 + (1+\lambda_1)(\mathbf{x}_2 + \mathbf{x}_3)\right) \\
1 &= \mathbf{y}_1 \cdot \mathbf{y}_1 \\
&= \frac{1}{3}\left((1-2\lambda_1)\mathbf{x}_1 + (1+\lambda_1)(\mathbf{x}_2 + \mathbf{x}_3)\right) \cdot \frac{1}{3}\left((1-2\lambda_1)\mathbf{x}_1 + (1+\lambda_1)(\mathbf{x}_2 + \mathbf{x}_3)\right) \\
&= \frac{1}{9}\left((1-2\lambda_1)^2 + 2(1+\lambda_1)^2 + 2(1-2\lambda_1)(1+\lambda_1)(\mathbf{x}_1\cdot\mathbf{x}_2 + \mathbf{x}_1\cdot\mathbf{x}_3) + 2(1+\lambda_1)^2\mathbf{x}_2\cdot\mathbf{x}_3\right) \\
\Rightarrow \quad 9 &= (1-2\lambda_1)^2 + 2(1+\lambda_1)^2 + 2(1-2\lambda_1)(1+\lambda_1)(\beta+\gamma) + 2(1+\lambda_1)^2\alpha \\
&= 3 + 6\lambda_1^2 + 2(\beta+\gamma) - 2(\beta+\gamma)\lambda_1 - 4\lambda_1^2(\beta+\gamma) + 2\alpha + 4\lambda_1\alpha + 2\lambda_1^2\alpha \\
0 &= (-6 + 2(\alpha+\beta+\gamma)) + 2(2\alpha - (\beta+\gamma))\lambda_1 + (6 + 2(\alpha - 2(\beta+\gamma)))\lambda_1^2 \\
\Rightarrow \quad 0 &= ((\alpha+\beta+\gamma) - 3) + (2\alpha - (\beta+\gamma))\lambda_1 + (3 + \alpha - 2(\beta+\gamma))\lambda_1^2 \\
&= (\lambda_1 + 1)((3 + \alpha - 2(\beta+\gamma))\lambda_1 + ((\alpha+\beta+\gamma) - 3)) \\
\Rightarrow \quad \lambda_1 &= \frac{3 - (\alpha+\beta+\gamma)}{3 + \alpha - 2(\beta+\gamma)}
\end{aligned}
$$

as required.
Since $\dfrac{GX_1}{GY_1} = \frac{1}{\lambda_1}$ we must have,

Question (2013 STEP III Q3)

The four vertices P_i $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ $(i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$. Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8\,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2\,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

Question (2014 STEP I Q7)

In the triangle OAB, the point D divides the side BO in the ratio $r : 1$ (so that $BD = rDO$), and the point E divides the side OA in the ratio $s : 1$ (so that $OE = sEA$), where r and s are both positive.

(i) The lines AD and BE intersect at G. Show that

$$g = \frac{rs}{1 + r + rs}\, a + \frac{1}{1 + r + rs}\, b\,,$$

where a, b and g are the position vectors with respect to O of A, B and G, respectively.

(ii) The line through G and O meets AB at F. Given that F divides AB in the ratio $t : 1$, find an expression for t in terms of r and s.

Question (2014 STEP III Q7)

The four distinct points P_i ($i = 1, 2, 3, 4$) are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q.

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

(ii) Let p_i be the position vector of the point P_i ($i = 1, 2, 3, 4$). Show that there exist numbers a_i, not all zero, such that

$$\sum_{i=1}^{4} a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} a_i p_i = \mathbf{0}\,. \qquad (*)$$

(iii) Let a_i ($i = 1, 2, 3, 4$) be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1p_1 + a_3p_3}{a_1 + a_3}\,.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$.

Question (2015 STEP II Q8)

xunit=1.0cm,yunit=1.0cm,algebraic=true,dimen=middle,dotstyle=o,dotsize=3pt 0,linewidth=0.3pt,arrowsize=3pt 2,arrowinset=0.25 (-2.94,-1.87)(7.07,3.86) (0,1)1.25 (3,0)0.55 [tl](5.33,-0.41)P (-2.44,-0.03)(6.18,-0.85) (-2.04,3.71)(6.55,-1.48) [tl](-0.18,1.1)C_1 [tl](2.85 ,0.15)C_2 [tl](-0.65,3.29)L' [tl](-1.5,-0.34)L

The diagram above shows two non-overlapping circles C_1 and C_2 of different sizes. The lines L and L' are the two common tangents to C_1 and C_2 such that the two circles lie on the same side of each of the tangents. The lines L and L' intersect at the point P which is called the *focus* of C_1 and C_2.

(i) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of the centres of C_1 and C_2, respectively. Show that the position vector of P is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2}\,,$$

where r_1 and r_2 are the radii of C_1 and C_2, respectively.

(ii) The circle C_3 does not overlap either C_1 or C_2 and its radius, r_3, satisfies $r_1 \neq r_3 \neq r_2$. The focus of C_1 and C_3 is Q, and the focus of C_2 and C_3 is R. Show that P, Q and R lie on the same straight line.

(iii) Find a condition on r_1, r_2 and r_3 for Q to lie half-way between P and R.

Question (2016 STEP I Q6)

The sides OA and CB of the quadrilateral $OABC$ are parallel. The point X lies on OA, between O and A. The position vectors of A, B, C and X relative to the origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$, respectively. Explain why $\mathbf{c}$ and $\mathbf{x}$ can be written in the form

$$\mathbf{c} = k\mathbf{a} + \mathbf{b} \qquad \text{and} \qquad \mathbf{x} = m\mathbf{a}\,,$$

where k and m are scalars, and state the range of values that each of k and m can take.

The lines OB and AC intersect at D, the lines XD and BC intersect at Y and the lines OY and AB intersect at Z. Show that the position vector of Z relative to O can be written as

$$\frac{\mathbf{b} + mk\mathbf{a}}{mk+1}\,.$$

The lines DZ and OA intersect at T. Show that

$$OT \times OA = OX \times TA \qquad \text{and} \qquad \frac{1}{OT} = \frac{1}{OX} + \frac{1}{OA}\,,$$

where, for example, OT denotes the length of the line joining O and T.

Question (2017 STEP II Q8)

All vectors in this question lie in the same plane.

The vertices of the non-right-angled triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to BC and CA, respectively.

Write down the vector equation of the line through A perpendicular to BC, in terms of $\mathbf{u}$, $\mathbf{a}$ and a parameter λ.

The line through A perpendicular to BC intersects the line through B perpendicular to CA at P. Find the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{u}$.

Hence show that the line CP is perpendicular to the line AB.

Question (2018 STEP II Q7)

The points O, A and B are the vertices of an acute-angled triangle. The points M and N lie on the sides OA and OB respectively, and the lines AN and BM intersect at Q. The position vector of A with respect to O is $\mathbf{a}$, and the position vectors of the other points are labelled similarly.

Given that $|MQ| = \mu|QB|$, and that $|NQ| = \nu|QA|$, where μ and ν are positive and $\mu\nu < 1$, show that

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\,\mathbf{a}\,.$$

The point L lies on the side OB, and $|OL| = \lambda|OB|$. Given that ML is parallel to AN, express λ in terms of μ and ν.

What is the geometrical significance of the condition $\mu\nu < 1$?

Question (2019 STEP I Q5) **(i)** The four points P, Q, R and S are the vertices of a plane quadrilateral. What is the geometrical shape of $PQRS$ if $\vec{PQ} = \vec{SR}$? What is the geometrical shape of $PQRS$ if $\vec{PQ} = \vec{SR}$ and $|\vec{PQ}| = |\vec{PS}|$?

(ii) A cube with edges of unit length has opposite vertices at $(0, 0, 0)$ and $(1, 1, 1)$. The points

$$P(p, 0, 0), \quad Q(1, q, 0), \quad R(r, 1, 1) \quad \text{and} \quad S(0, s, 1)$$

lie on edges of the cube. Given that the four points lie in the same plane, show that

$$rq = (1 - s)(1 - p).$$

a) Show that $\vec{PQ} = \vec{SR}$ if and only if the centroid of the quadrilateral $PQRS$ is at the centre of the cube. Note: the centroid of the quadrilateral $PQRS$ is the point with position vector

$$\frac{1}{4}(\vec{OP} + \vec{OQ} + \vec{OR} + \vec{OS}),$$

where O is the origin.

b) Given that $\vec{PQ} = \vec{SR}$ and $|\vec{PQ}| = |\vec{PS}|$, express q, r and s in terms of p. Show that

$$\cos PQR = \frac{4p - 1}{5 - 4p + 8p^2}.$$

Write down the values of p, q, r and s if $PQRS$ is a square, and show that the length of each side of this square is greater than $\frac{21}{20}$.

(i) If $\vec{PQ} = \vec{SR}$ we have a parallelogram.

$\vec{PQ} = \vec{SR}$ and $|\vec{PQ}| = |\vec{PS}|$ then we have a rhombus.

(ii) If the four points lie in a plane then

$(\vec{RS} \times \vec{RP}) \cdot \vec{RQ} = 0$, so

$$\begin{aligned}
0 &= \left(\begin{pmatrix} -r \\ s-1 \\ 0 \end{pmatrix} \times \begin{pmatrix} p-r \\ -1 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1-r \\ q-1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1-s \\ -r \\ r + (p-r)(1-s) \end{pmatrix} \cdot \begin{pmatrix} 1-r \\ q-1 \\ -1 \end{pmatrix} \\
&= (1-s)(1-r) - r(q-1) - r - (p-r)(1-s) \\
&= (1-s)(1-r-p+r) - rq \\
\Rightarrow \quad rq &= (1-s)(1-p)
\end{aligned}$$

a)

$\Leftrightarrow 1 = r + p \quad ; \quad 1 = q + s$

The centroid is $\frac{1}{4}(p+1+r, q+s+1, 2)$ which is clearly $\frac{1}{2}(1,1,1)$ iff those equations are true.

b)

$$\begin{aligned} & & |\vec{PQ}| = |\vec{PS}| \\ &\Leftrightarrow & (1-p)^2 + q^2 + 0^2 = p^2 + s^2 + 1) \\ &\Leftrightarrow & 1 - 2p + p^2 + q^2 = p^2 + s^2 + 1 \\ &\Leftrightarrow & -2p + q^2 = s^2 \end{aligned}$$

From the previous equations we have $r = 1 - p$, and $-2p + (1-s)^2 = s^2 \Rightarrow -2p + 1 - 2s = 0 \Rightarrow s = \frac{1}{2} - p$ and $q = \frac{1}{2} + p$

$$\begin{aligned} \cos PQR &= \frac{\vec{QP} \cdot \vec{QR}}{|\vec{QP}||\vec{QR}|} \\ &= \frac{\begin{pmatrix} p-1 \\ -q \\ 0 \end{pmatrix} \cdot \begin{pmatrix} r-1 \\ 1-q \\ 1 \end{pmatrix}}{\sqrt{(p-1)^2 + q^2}\sqrt{(r-1)^2 + (1-q)^2 + 1^2}} \\ &= \frac{\begin{pmatrix} p-1 \\ -\frac{1}{2} - p \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -p \\ \frac{1}{2} - p \\ 1 \end{pmatrix}}{\sqrt{(p-1)^2 + (-\frac{1}{2} - p)^2}\sqrt{p^2 + (\frac{1}{2} - p)^2 + 1^2}} \\ &= \frac{p - p^2 - \frac{1}{4} + p^2}{\sqrt{p^2 - 2p + 1 + \frac{1}{4} + p + p^2}\sqrt{p^2 + \frac{1}{4} - p + p^2 + 1}} \\ &= \frac{4p - 1}{\sqrt{8p^2 - 4p + 5}\sqrt{8p^2 - 4p + 5}} \\ &= \frac{4p - 1}{8p^2 - 4p + 5} \end{aligned}$$

For $PQRS$ to be a square $\cos PQR = 0$, ie $p = \frac{1}{4}$ and so $(p, q, r, s) = (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})$ and $|PQ| = \sqrt{(1-p)^2 + q^2} = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{2}}{4}$, notice that $\left(\frac{21}{20}\right)^2 = \frac{441}{400} < \frac{9}{8}$ $(441 < 450)$ therefore the sides are at least as long as $\frac{21}{20}$

Question (1989 STEP II Q9)

The matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n,$$

where $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$ and t is a variable scalar. Evaluate $\mathbf{A}^2$, and show that

$$\mathbf{F} = \mathbf{I}\cosh t + \mathbf{A}\sinh t.$$

Show also that $\mathbf{F}^{-1} = \mathbf{I}\cosh t - \mathbf{A}\sinh t$, and that $\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{FA}$. The vector $\mathbf{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{Ar} = \mathbf{0},$$

with $x = \alpha$ and $y = \beta$ at $t = 0$. Solve this equation by means of a suitable matrix integrating factor, and hence show that

$$\begin{aligned} x(t) &= \alpha\cosh t + (3\alpha + \beta)\sinh t \\ y(t) &= \beta\cosh t - (8\alpha + 3\beta)\sinh t. \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}^2 &= \begin{pmatrix} 9-8 & 3-3 \\ -24+24 & -8+9 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I} \end{aligned}$$

Therefore:

$$\begin{aligned} \mathbf{F} &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n \\ &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} t^{2n}\mathbf{I} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} t^{2n+1}\mathbf{A} \\ &= \cosh t\mathbf{I} + \sinh t\mathbf{A} \end{aligned}$$

Notice that

$$\begin{aligned} \mathbf{F}(\mathbf{I}\cosh t - \mathbf{A}\sinh t) &= (\mathbf{I}\cosh t + \mathbf{A}\sinh t)(\mathbf{I}\cosh t - \mathbf{A}\sinh t) \\ &= \mathbf{I}^2\cosh^2 t + \mathbf{A}(\sinh t\cosh t - \cosh t\sinh t) - \mathbf{A}^2\sinh^2 t \\ &= \mathbf{I}\cosh^2 t - \mathbf{I}\sinh^2 t \\ &= \mathbf{I} \end{aligned}$$

Therefore $\mathbf{F}^{-1} = \mathbf{I}\cosh t - \mathbf{A}\sinh t$

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n\right]$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} \mathbf{A}^n \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} \mathbf{A}^{n-1} \right) \mathbf{A} \\
&= \mathbf{FA}
\end{aligned}$$

$$\begin{aligned}
& & \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{Ar} &= \mathbf{0} \\
&\Rightarrow & \mathbf{F}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{FAr} &= \mathbf{0} \\
& & \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{Fr}) &= 0 \\
&\Rightarrow & \mathbf{Fr} &= \mathbf{c} \\
&\Rightarrow & \mathbf{r} &= \mathbf{F}^{-1}\mathbf{c} \\
& & &= (\mathbf{I}\cosh t - \mathbf{A}\sinh t)\mathbf{c} \\
t = 0: & & \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \mathbf{c} \\
&\Rightarrow & \mathbf{r} &= (\mathbf{I}\cosh t - \mathbf{A}\sinh t)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
& & &= \begin{pmatrix} \alpha\cosh t \\ \beta\cosh t \end{pmatrix} - \begin{pmatrix} -3\alpha - \beta \\ 8\alpha + 3\beta \end{pmatrix} \sinh t \\
& & &= \begin{pmatrix} \alpha\cosh t + (3\alpha+\beta)\sinh t \\ \beta\cosh t - (8\alpha+3\beta)\sinh t \end{pmatrix}
\end{aligned}$$

as required

Question (1990 STEP I Q7)

Let y, u, v, P and Q all be functions of x. Show that the substitution $y = uv$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

leads to an equation for $\frac{\mathrm{d}v}{\mathrm{d}x}$ in terms of x, Q and u, provided that u satisfies a suitable first order differential equation. Hence or otherwise solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}},$$

given that $y(1) = 0$. For what set of values of x is the solution valid?

Suppose $y = uv$ then and suppose $\frac{\mathrm{d}u}{\mathrm{d}x} + Pu = 0$ then

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + Py &= Q \\
uv' + u'v + Puv &= Q \\
uv' &= Q \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{Q}{u}
\end{aligned}$$

Consider

$$\begin{aligned} 0 &= \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2u}{x+1} \\ \Rightarrow \qquad \ln u &= 2\ln(1+x) + C \\ \Rightarrow \qquad u &= A(1+x)^2 \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{A}(x+1)^{\frac{1}{2}} \\ \Rightarrow \qquad v &= \frac{2}{3A}(x+1)^{\frac{3}{2}} + k \\ \Rightarrow \qquad y &= \frac{2}{3}(x+1)^{\frac{7}{2}} + k(x+1)^2 \\ 0 &= y(1) \\ &= \frac{2}{3}2^{7/2} + k2^2 \\ \Rightarrow \qquad k &= -\frac{2^{5/2}}{3} \\ \Rightarrow \qquad y &= \frac{2}{3}(x+1)^{7/2} - \frac{2^{5/2}}{3}(x+1)^2 \end{aligned}$$

Question (1990 STEP III Q8)

Let P, Q and R be functions of x. Prove that, for any function y of x, the function

$$Py'' + Qy' + Ry$$

can be written in the form $\dfrac{\mathrm{d}}{\mathrm{d}x}(py' + qy)$, where p and q are functions of x, if and only if $P'' - Q' + R = 0$. Solve the differential equation

$$(x - x^4)y'' + (1 - 7x^3)y' - 9x^2y = (x^3 + 3x^2)\mathrm{e}^x,$$

given that when $x = 2, y = 2\mathrm{e}^2$ and $y' = 0$.

Suppose $Py'' + Qy' + Ry = \frac{\mathrm{d}}{\mathrm{d}x}(py' + qy)$, then

$$\begin{aligned} Py'' + Qy' + Ry &= \frac{\mathrm{d}}{\mathrm{d}x}(py' + qy) \\ &= py'' + p'y' + qy' + q'y \\ &= py'' + (p' + q)y' + q'y \end{aligned}$$

Therefore $P = p, Q = p' + q, R = q'$, Therefore $q = Q - P'$ and $R = Q' - P''$ or $P'' - Q' + R = 0$.

($\Rightarrow$) Suppose it can be written in that form, then the logic we have applied shows that equation is true. ($\Leftarrow$) Suppose $P'' - Q' + R = 0$, then letting $p = P, q = Q - P'$ we find functions of the form which will be expressed correctly.

Notice that if $P = x - x^4, Q = (1 - 7x^3), R = -9x^2$ then:

$$P'' - Q' + R = -12x^2 + 21x^2 - 9x^2$$

$$= 0$$

Therefore we can write our second order ODE as:

$$\begin{aligned}(x^3 + 3x)\mathrm{e}^x &= \frac{\mathrm{d}}{\mathrm{d}x}\big((x - x^4)y' + (1 - 7x^3 - (1 - 4x^3))y\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\big((x - x^4)y' - 3x^3 y\big)\end{aligned}$$

Suppose $z = (x - x^4)y' - 3x^2 y$, then $z = (2 - 2^4) \cdot 0 - 3 \cdot 2^2 \cdot 2e^2 = -24e^2$ when $x = 2$. and we have:

$$\begin{aligned} && \frac{\mathrm{d}z}{\mathrm{d}x} &= (x^3 + 3x^2)e^x \\ \Rightarrow && z &= \int (x^3 + 3x^2)e^x \mathrm{d}x \\ && &= x^3 e^x + c \\ \Rightarrow && -48e^2 &= e^2(8) + c \\ \Rightarrow && c &= -56e^2 \\ \Rightarrow && z &= e^x(x^3) - 56e^2 \end{aligned}$$

So our differential equation is:

$$\begin{aligned} && (x - x^4)y' - 3x^3 y &= x^3 e^x - 56e^2 \\ \Rightarrow && (1 - x^3)y' - 3x^2 y &= x^2 e^x - \frac{6e^2}{x} \\ \Rightarrow && \frac{\mathrm{d}}{\mathrm{d}x}\big((1 - x^3)y\big) &= x^2 e^x - \frac{56e^2}{x} \\ \Rightarrow && (1 - x^3)y &= (x^2 - 2x + 2)e^x - 56e^2 \ln x + k \\ \underbrace{\Rightarrow}_{x=2} && (1 - 2^3)2e^2 &= (2^2 - 2 \cdot 2 + 2)e^2 - 56e^2 \ln 2 + k \\ \Rightarrow && k &= -16e^2 + 56 \ln 2 \cdot e^2 \\ \Rightarrow && y &= \frac{(x^2 - 2x + 2)e^x - 56e^2 \ln x - 16e^2 + 56 \ln 2 \cdot e^2}{(1 - x^3)} \end{aligned}$$

Question (1995 STEP II Q8)

If there are x micrograms of bacteria in a nutrient medium, the population of bacteria will grow at the rate $(2K - x)x$ micrograms per hour. Show that, if $x = K$ when $t = 0$, the population at time t is given by

$$x(t) = K + K\frac{1 - \mathrm{e}^{-2Kt}}{1 + \mathrm{e}^{-2Kt}}.$$

Sketch, for $t \geqslant 0$, the graph of x against t. What happens to $x(t)$ as $t \to \infty$?

Now suppose that the situation is as described in the first paragraph, except that we remove the bacteria from the nutrient medium at a rate L micrograms per hour where $K^2 > L$. We set $\alpha = \sqrt{K^2 - L}$. Write down the new differential equation for x. By considering a new variable $y = x - K + \alpha$, or otherwise, show that, if $x(0) = K$ then $x(t) \to K + \alpha$ as $t \to \infty$.

Question (2000 STEP II Q8) **(i)** Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\mathrm{e}^{-x^2}(y+3)^{\frac{1}{2}} = 0 \qquad (x \geq 0),$$

that satisfies the condition $y = 6$ when $x = 0$. Find y in terms of x and show that $y \to 1$ as $x \to \infty$.

(ii) Let y be any solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x\mathrm{e}^{6x^2}(y+3)^{1-k} = 0 \qquad (x \geq 0).$$

Find a value of k such that, as $x \to \infty$, $\mathrm{e}^{-3x^2}y$ tends to a finite non-zero limit, which you should determine.

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ may be assumed.]

Question (2003 STEP III Q8) **(i)** Show that the gradient at a point $(x\,,\,y)$ on the curve

$$(y + 2x)^3\,(y - 4x) = c\,,$$

where c is a constant, is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16x - y}{2y - 5x}\,.$$

(ii) By considering the derivative with respect to x of $(y + ax)^n\,(y + bx)$, or otherwise, find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{10x - 4y}{3x - y}\,.$$

(i)

$$
\begin{aligned}
& c = (y+2x)^3(y-4x) \\
\Rightarrow\quad & 0 = 3(y+2x)^2(y-4x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+2\right) + (y+2x)^3\left(\frac{\mathrm{d}y}{\mathrm{d}x}-4\right) \\
\Rightarrow\quad & 0 = 3(y-4x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+2\right) + (y+2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}-4\right) \\
\Rightarrow\quad & = \frac{\mathrm{d}y}{\mathrm{d}x}(3(y-4x)+(y+2x)) + 6(y-4x) - 4(y+2x) \\
& = \frac{\mathrm{d}y}{\mathrm{d}x}(4y-10x) + 2y - 32x \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16x-y}{2y-5x}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
& c = (y+ax)^n(y+bx) \\
\Rightarrow\quad & 0 = n(y+ax)^{n-1}(y+bx)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+a\right) + (y+ax)^n\left(\frac{\mathrm{d}y}{\mathrm{d}x}+b\right) \\
\Rightarrow\quad & 0 = n(y+bx)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+a\right) + (y+ax)\left(\frac{\mathrm{d}y}{\mathrm{d}x}+b\right) \\
& = \frac{\mathrm{d}y}{\mathrm{d}x}((n+1)y+(nb+a)x) + an(y+bx) + by + bax \\
& = \frac{\mathrm{d}y}{\mathrm{d}x}((n+1)y+(nb+a)x) + (an+b)y + ab(n+1)x \\
\Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(an+b)y+ab(n+1)x}{(n+1)y+(nb+a)x}
\end{aligned}
$$

We must have $ab = 10, a+b = -7$ so say $a = -5, b = -2, n = 2$ and we have

$(y-5x)^2(y-2) = c$ is our general solution to the differential equation

Question (2004 STEP III Q8)
Show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + \frac{g(x)}{y}$$

then the substitution $u = y^2$ gives a linear differential equation for $u(x)$. Hence or otherwise solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - \frac{1}{y}\,.$$

Determine the solution curves of this equation which pass through $(1\,,1)$, $(2\,,2)$ and $(4\,,4)$ and sketch graphs of all three curves on the same axes.

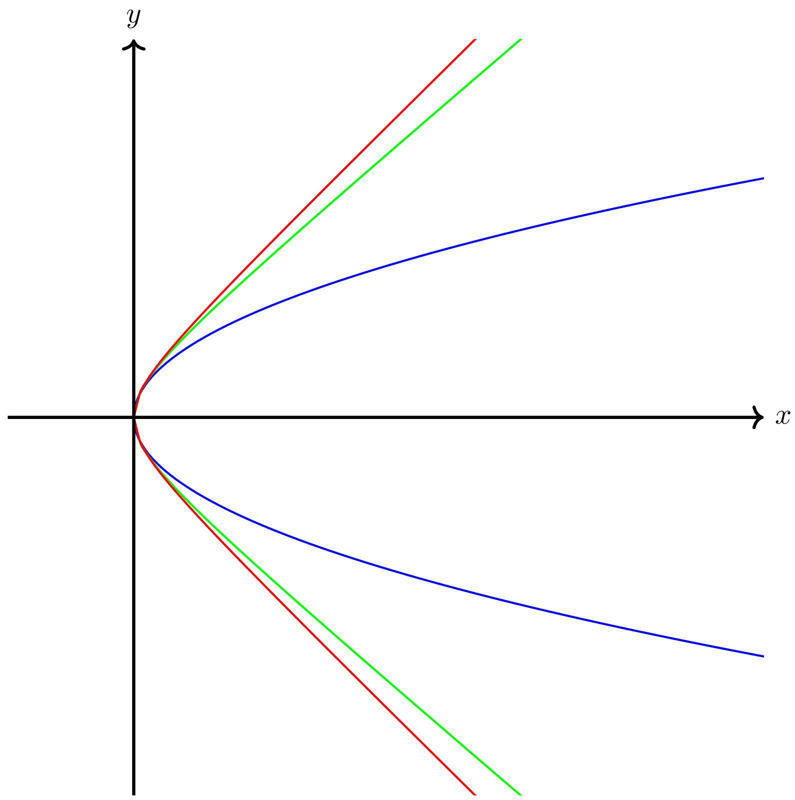
$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)y + \frac{g(x)}{y} \\
y\frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)y^2 + g(x)
\end{aligned}
$$

$$u = y^2 : \qquad \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)u + g(x)$$

Which is a linear differential equation for u.

$$\begin{aligned}
& & \frac{1}{2}u' &= \frac{1}{x}u - 1 \\
&\Rightarrow & u' - \frac{2}{x}u &= -1 \\
&\Rightarrow & \frac{1}{x^2}u' - \frac{2}{x^3}u &= -\frac{1}{x^2} \\
&\Rightarrow & (\frac{u}{x^2})' &= -\frac{1}{x^2} \\
&\Rightarrow & \frac{u}{x^2} &= \frac{1}{x} + C \\
&\Rightarrow & u &= Cx^2 + x \\
&\Rightarrow & y^2 &= Cx^2 + x
\end{aligned}$$

If $(1,1)$ is on the curve then $1 = C + 1 \Rightarrow C = 0 \Rightarrow y^2 = x$.
If $(2,2)$ is on the curve then $4 = 4C + 2 \Rightarrow C = \frac{1}{2} \Rightarrow y^2 = x + \frac{1}{2}x^2$.
If $(3,3)$ is on the curve then $9 = 9C + 3 \Rightarrow C = \frac{2}{3} \Rightarrow y^2 = x + \frac{2}{3}x^2$



Question (2012 STEP I Q8) **(i)** Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \quad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \quad (x \neq 0).$$

(i)

$$\begin{aligned}
& y = xv \\
& y' = v + xv' \\
\Rightarrow \quad & 0 = x^2v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2v^2) - 2x^2 \\
& \quad = 2x^2v^2 + x^3v\frac{\mathrm{d}v}{\mathrm{d}x} - 2x^2 \\
\Rightarrow \quad & 0 = xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & \frac{v}{1-v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2}{x} \\
\Rightarrow \quad & \int \frac{v}{1-v^2}\mathrm{d}v = 2\ln|x| \\
\Rightarrow \quad & -\frac{1}{2}\ln|1-v^2| = 2\ln|x| + C \\
\Rightarrow \quad & 4\ln|x| + \ln|1-v^2| = K \\
\Rightarrow \quad & x^4(1-v^2) = K \\
\Rightarrow \quad & x^2(x^2-y^2) = K
\end{aligned}$$

(ii)

Question (2014 STEP II Q5)

Given that $y = xu$, where u is a function of x, write down an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y+x}{y-2x}$$

given that the solution curve passes through the point $(1, 1)$. Give your answer in the form of a quadratic in x and y.

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b, or otherwise, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-2y-4}{2x+y-3},$$

given that the solution curve passes through the point $(1, 1)$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(y) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(xu) \\
&\underbrace{=}_{\text{product rule}} \frac{\mathrm{d}}{\mathrm{d}x}(x)\,u + x\frac{\mathrm{d}}{\mathrm{d}x}(u) \\
&= u + x\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

(i)

$$\begin{aligned}
&& \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2y+x}{y-2x} \\
&& u + x\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{2u+1}{u-2} \\
&& x\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{2u-1-u^2+2u}{u-2} \\
\Rightarrow && \int \frac{2-u}{u^2-4u+1}\,\mathrm{d}u &= \int \frac{1}{x}\,\mathrm{d}x \\
&& \int \frac{2-u}{(u-2)^2-5}\,\mathrm{d}u &= \int \frac{1}{x}\,\mathrm{d}x \\
&& -\frac{1}{2}\ln|(u-2)^2-5| &= \ln x + C \\
(x,y)=(1,1): && -\ln 2 &= C \\
\Rightarrow && \ln x^2 &= \ln 4 - \ln|5-(u-2)^2| \\
\Rightarrow && x^2 &= \frac{4}{5-(u-2)^2} \\
\Rightarrow && 4 &= x^2\left(5 - \left(\frac{y}{x}-2\right)^2\right) \\
&& &= 5x^2 - (y-2x)^2 \\
&& &= x^2 + 4xy - y^2
\end{aligned}$$

(ii) It would be convienient if $x-2y-4 = X-2Y$ and $2x+y-3 = 2X+Y$, ie $a-2b = 4$ and $2a+b = 3$, ie $a = 2, b = -1$.

Now our differential equation is:

$$\begin{aligned}\frac{\mathrm{d}Y}{\mathrm{d}X} &= \frac{X-2Y}{2X+Y} \\ \frac{\mathrm{d}X}{\mathrm{d}Y} &= \frac{2X+Y}{X-2Y}\end{aligned}$$

This is the same differential equation we have already solved, just with the roles of x and y interchanged with Y and X and with the point $(0,3)$ being on the curve, ie:

$Y^2 + 4XY - X^2 = c$ and $c = 9$, therefore our equation is:

$$(y-1)^2 + 4(y-1)(x+2) - (x+2)^2 = 9$$

Question (2018 STEP II Q8)**(i)** Use the substitution $v = \sqrt{y}$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{1}{2}} - \beta y \qquad (y \geq 0, \quad t \geq 0)\,,$$

where α and β are positive constants. Find the non-constant solution $y_1(x)$ that satisfies $y_1(0) = 0$.

(ii) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{2}{3}} - \beta y \qquad (y \geq 0, \quad t \geq 0)\,,$$

where α and β are positive constants. Find the non-constant solution $y_2(x)$ that satisfies $y_2(0) = 0$.

(iii) In the case $\alpha = \beta$, sketch $y_1(x)$ and $y_2(x)$ on the same axes, indicating clearly which is $y_1(x)$ and which is $y_2(x)$. You should explain how you determined the positions of the curves relative to each other.

Question (1987 STEP I Q3)

By substituting $y(x) = xv(x)$ in the differential equation

$$x^3\frac{\mathrm{d}v}{\mathrm{d}x} + x^2 v = \frac{1+x^2v^2}{(1+x^2)\,v},$$

or otherwise, find the solution $v(x)$ that satisfies $v = 1$ when $x = 1$.

What value does this solution approach when x becomes large?

Let $y = xv$ then $y' = v + xv'$ and so $x^2y' = x^2v + x^3v'$ Our differential equation is now:

$$\begin{aligned}
 && x^2y' &= \frac{1+y^2}{(1+x^2)\frac{y}{x}} \\
\Rightarrow && xy' &= \frac{(1+y^2)}{(1+x^2)y} \\
\Rightarrow && \frac{y}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x(1+x^2)} \\
\Rightarrow && \frac{y}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x} - \frac{x}{1+x^2} \\
\Rightarrow && \frac12 \ln(1+y^2) &= \ln x - \frac12 \ln(1+x^2) + C \\
\Rightarrow && \frac12 \ln(1+y^2) &= \frac12 \ln \left(\frac{x^2}{1+x^2} \right) + C \\
\Rightarrow && 1+y^2 &= \frac{Dx^2}{1+x^2} \\
\Rightarrow && D &= 4 \qquad : (x=1, v=1, y=1) \\
\Rightarrow && 1+x^2v^2 &= \frac{4x^2}{1+x^2} \\
\Rightarrow && v^2 &= \frac{3x^2-1}{x^2(1+x^2)} \\
\Rightarrow && v &= \sqrt{\frac{3x^2-1}{x^2(1+x^2)}}
\end{aligned}$$

As $x \to \infty$, $v \to 0$

Question (1988 STEP II Q5)

By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully. Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

Suppose $z^7 = 1$, then we can write $z = \cos\theta + i\sin\theta$ and we must have that:

$$
\begin{aligned}
0 &= \operatorname{Im}((\cos\theta + i\sin\theta)^7) \\
&= \binom{7}{6}\cos^6\theta\sin\theta - \binom{7}{4}\cos^4\theta\sin^3\theta + \binom{7}{2}\cos^2\theta\sin^5\theta - \sin^7\theta \\
&= 7\cos^6\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta \\
&= -\cos^7\theta\left(\tan^7\theta - 21\tan^5\theta + 35\tan^3\theta - 7\tan\theta\right) \\
&= \cos^7\theta \cdot t(t^7 - 21t^4 + 35t^2 - 7)
\end{aligned}
$$

Where $t = \tan\theta$. So if z is a root of $z^7 = 1$ and $\cos\theta \neq 0, \tan\theta \neq 0$ then t is a root of the equation. Thererefore the roots are:

$\tan\frac{2\pi k}{7}$ where $k = 1, 2, \dots 6$.

Noting that $\tan\frac{\pi}{7} = -\tan\frac{6\pi}{7}, \tan\frac{3\pi}{7} = -\tan\frac{4\pi}{7}, \tan\frac{5\pi}{7} = -\tan\frac{2\pi}{7}$ we can conclude that:

$$
\begin{aligned}
7 &= \prod_{k=1}^{k}\tan\frac{k\pi}{6} \\
&= \left(\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7}\right)^2 \\
\Rightarrow \qquad \pm\sqrt{7} &= \tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7}
\end{aligned}
$$

However, we know that $\tan\frac{2\pi}{7}$ is positive, $\tan\frac{4\pi}{7}, \tan\frac{6\pi}{7}$ are negative, therefore the result must be positive, ie $+\sqrt{7}$

Using a similar method, we notice that:

$$
\begin{aligned}
0 &= \operatorname{Im}\left((\cos\theta + i\sin\theta)^n\right) \\
&= \cos^n\theta \cdot t(t^{n-1} + \cdots - \binom{n}{n-1})
\end{aligned}
$$

Therefore $\prod_{k=0}^{n-1}\tan\frac{k\pi}{n} = n$ and since $\tan\frac{(2k+1)\pi}{n} = \tan\frac{(n-2k-1)\pi}{n}$ is a map of all the odd numbers to the even numbers (and vice versa) when n is odd. We also know that the terms less where $\tan\theta$ has $\theta < \frac{\pi}{2}$ are positive, and the others even, we can determine the signs:

$$
\begin{aligned}
\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}\tan\frac{6\pi}{9}\tan\frac{8\pi}{9} &= 3 \\
\tan\frac{2\pi}{11}\tan\frac{4\pi}{11}\tan\frac{6\pi}{11}\tan\frac{8\pi}{11}\tan\frac{10\pi}{11} &= -\sqrt{11}
\end{aligned}
$$

Question (1989 STEP I Q8)

By using de Moivre's theorem, or otherwise, show that

(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$;

(ii) $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

Hence, or otherwise, find all the real roots of the equation

$$16x^6 - 28x^4 + 13x^2 - 1 = 0.$$

[No credit will be given for numerical approximations.]

Given that $e^{i\theta} = \cos\theta + i\sin\theta$ we must have that

(i)

$$\begin{aligned}
\cos 4\theta &= \text{Re}\left(e^{i4\theta}\right) \\
&= \text{Re}\left((\cos\theta + i\sin\theta)^4\right) \\
&= \cos^4\theta - \binom{4}{2}\cos^2\theta\sin^2\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

(ii) Similarly,

$$\begin{aligned}
\cos 6\theta &= \text{Re}\left(e^{i6\theta}\right) \\
&= \text{Re}\left((\cos\theta + i\sin\theta)^6\right) \\
&= \cos^6\theta - \binom{6}{2}\cos^4\theta\sin^2\theta + \binom{6}{4}\cos^2\theta\sin^4\theta - \sin^6\theta \\
&= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3 \\
&= 31\cos^6\theta - 45\cos^4\theta + 15\cos^2\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta \\
&= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1
\end{aligned}$$

$$\begin{aligned}
0 &= 16x^6 - 28x^4 + 13x^2 - 1 \\
&= \frac{1}{2}(32x^6 - 56x^4 + 26x^2 - 1) \\
&= \frac{1}{2}(32x^6 - 48x^4 + 18x^2 - 1 - (8x^4 - 8x^2 + 1))
\end{aligned}$$

Therefore if $x = \cos\theta$ then we are looking at solving $\cos 6\theta = \cos 4\theta$.

$\cos 6\theta - \cos 4\theta = -2\sin 5\theta \sin\theta = 0$. So we should be looking at $\sin 5\theta = 0$ and $\sin\theta = 0$.

$\sin\theta = 0 \Rightarrow x = \cos\theta = \pm 1$ both of which are roots.

The other roots will be $\cos\frac{\pi}{5}, \cos\frac{2\pi}{5}$ etc but it's unclear this is an acceptable form.

Alternatively, given our two roots, we can factorize

$$\begin{aligned} 0 &= 16x^6 - 28x^4 + 13x^2 - 1 \\ &= (x^2-1)(16x^4 - 12x^2 + 1) \end{aligned}$$

We can solve $16y^2 - 12y + 1 = 0$ to see that $x^2 = \frac{3\pm\sqrt{5}}{8}$ so our roots are:

$x = -1, 1, \pm\sqrt{\frac{3+\sqrt{5}}{8}}, \pm\sqrt{\frac{3-\sqrt{5}}{8}}$

(We might notice that $3+\sqrt{5} = \left(\frac{1+\sqrt{5}}{\sqrt{2}}\right)^2$ so our final answer could be: $x = -1, 1, \pm\frac{1+\sqrt{5}}{4}, \pm\frac{\sqrt{5}-1}{4}$)

Question (1990 STEP I Q2)

Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$. Show that $1+\omega+\omega^2 = 0$ and calculate the modulus and argument of $1+\omega^2$. Let n be a positive integer. By evaluating $(1+\omega^r)^n$ in two ways, taking $r = 1, 2$ and 3, or otherwise, prove that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{k} = \frac{1}{3}\left(2^n + 2\cos\left(\frac{n\pi}{3}\right)\right),$$

where k is the largest multiple of 3 less than or equal to n. Without using a calculator, evaluate

$$\binom{25}{0} + \binom{25}{3} + \cdots + \binom{25}{24}$$

and

$$\binom{24}{2} + \binom{24}{5} + \cdots + \binom{24}{23}.$$

[$2^{25} = 33554432$.]

Since $\omega^3 = 1$ and $\omega \neq 1$ we must have that $(\omega-1)(1+\omega+\omega^2) = 0$ but by dividing by $\omega - 1$ we obtain the desired result.

$1+\omega^2 = -\omega$ so $|1+\omega^2| = |-\omega| = 1$ and $\arg(1+\omega^2) = \arg(-\omega) = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$$\begin{aligned} (1+1)^n &= \sum_{k=0}^{n}\binom{n}{k} \\ (1+\omega)^n &= \sum_{k=0}^{n}\binom{n}{k}\omega^k \\ (1+\omega^2)^n &= \sum_{k=0}^{n}\binom{n}{k}\omega^{2k} \end{aligned}$$

$$\Rightarrow \quad 2^n + (-\omega^2)^n + (-\omega)^n = \sum_{k=0, k\equiv 0 \pmod 3}^{n} (1+1+1)\binom{n}{k} + \sum_{k=0, k\equiv 1 \pmod 3}^{n} (1+\omega+\omega^2)\binom{n}{k} +$$

$$\Rightarrow \quad 2^n + ((-\omega)^n)^{-1} + (-\omega)^n = \sum_{k=0, k\equiv 0 \pmod 3}^{n} \binom{n}{k}$$

$2^n + ((-\omega)^n)^{-1} + (-\omega)^n = 2^n + 2\text{Re}(-\omega^n) = 2^n + 2\cos\frac{n\pi}{3}$

Therefore our answer follows.

$$\begin{aligned}\binom{25}{0} + \binom{25}{3} + \cdots + \binom{25}{24} &= \frac{1}{3}\left(2^{25} + 2\cos(\frac{25\pi}{3})\right) \\ &= \frac{1}{3}\left(2^{25} + 2\cos\frac{\pi}{3}\right) \\ &= \frac{1}{3}\left(2^{25} + 1\right) \\ &= \frac{1}{3}\left((4096 \cdot 4096 \cdot 2) + 1\right) \\ &= 11\,184\,811\end{aligned}$$

Notice that $S_2 = \binom{24}{2} + \cdots + \binom{24}{23} = \binom{24}{1} + \cdots + \binom{24}{22} = S_1$ and $S_0 = \binom{24}{0} + \cdots + \binom{24}{21} = \frac{1}{3}\left(2^{24} + 2\right)$

Therefore since $S_0 + 2 \cdot S_2 = 2^{24}$ we must have

$$\begin{aligned}S_2 &= \frac{1}{2}\left(2^{24} - \frac{1}{3}\left(2^{24} + 2\right)\right) \\ &= \frac{1}{3}\left(2^{24} - 1\right) \\ &= \frac{1}{3}\left(16777216 - 1\right) \\ &= \frac{1}{3} \cdot 16777215 \\ &= 5\,592\,405\end{aligned}$$

Question (1990 STEP III Q1)

Show, using de Moivre's theorem, or otherwise, that

$$\tan 9\theta = \frac{t(t^2-3)(t^6-33t^4+27t^2-3)}{(3t^2-1)(3t^6-27t^4+33t^2-1)}, \qquad \text{where } t = \tan\theta.$$

By considering the equation $\tan 9\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{9}\right), \qquad \tan^2\left(\frac{2\pi}{9}\right) \qquad \text{and} \qquad \tan^2\left(\frac{4\pi}{9}\right).$$

Deduce the value of

$$\tan\left(\frac{\pi}{9}\right)\tan\left(\frac{2\pi}{9}\right)\tan\left(\frac{4\pi}{9}\right).$$

Show that

$$\tan^6\left(\frac{\pi}{9}\right) + \tan^6\left(\frac{2\pi}{9}\right) + \tan^6\left(\frac{4\pi}{9}\right) = 33273.$$

Writing $c = \cos\theta, s = \sin\theta$ then de Moivre states that:

$$\begin{aligned}
\cos 9\theta + i\sin 9\theta &= (c+is)^9 \\
&= c^9 + 9ic^8s - 36c^7s^2 - 84ic^6s^3 + 126c^5s^4 + 126ic^4s^5 - 84c^3s^6 - 36ic^2s^7 + 9cs^8 + \\
&= (c^9 - 36c^7s^2 + 126c^5s^3 - 84c^3s^6 + 8cs^8) + i(9c^8s - 75c^6s^3 + 126c^4s^5 - 36c^2s^7 + \\
\Rightarrow \qquad \tan 9\theta &= \frac{(9c^8s - 75c^6s^3 + 126c^4s^5 - 36s^2c^7 + s^9)}{(c^9 - 36c^7s^2 + 126c^5s^4 - 84c^3s^6 + 8cs^8)} \\
&= \frac{9t - 75t^3 + 126s^5 - 36t^7 + t^9}{1 - 36t^2 + 126t^4 - 84t^6 + 8t^8} \\
&= \frac{t(t^2-3)(t^6-33t^4+27t^2-3)}{(3t^2-1)(3t^6-27t^4+33t^2-1)}
\end{aligned}$$

If we consider $\tan 9\theta = 0$ it will have the roots $\theta = \frac{n\pi}{9}, n \in \mathbb{Z}$, in particular, the numerator of our fraction for $\tan 9\theta$ will be zero for $t = 0, \tan\frac{\pi}{9}, \tan\frac{2\pi}{9}, \tan\frac{3\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{5\pi}{9}, \tan\frac{6\pi}{9}, \tan\frac{7\pi}{9}, \tan$ It's worth noting all other values of θ will repeat these values. Also note that $0, \tan\frac{\pi}{3}, \tan\frac{2\pi}{3}$ are the roots of t and $t^2 - 3$ respectively. Therefore the other values are the roots of our sextic. However, also note that $\tan\frac{8\pi}{9} = -\tan\frac{\pi}{9}$ and similar, therefore we can notice that all the roots in pairs can be mapped to $\tan\frac{\pi}{9}, \tan\frac{2\pi}{9}$ and $\tan\frac{4\pi}{9}$ and all those values are squared, so the roots of:

$x^3 - 33x^2 + 27x - 3$ will be $\tan^2\frac{\pi}{9}, \tan^2\frac{2\pi}{9}$ and $\tan^2\frac{4\pi}{9}$.

The product of the roots will be 3, so

$$\begin{aligned}
\tan^2\frac{\pi}{9}\tan^2\frac{2\pi}{9}\tan^2\frac{4\pi}{9} &= 3 \\
\Rightarrow \qquad \tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{4\pi}{9} &= \pm\sqrt{3}
\end{aligned}$$

$$\underbrace{\Rightarrow}_{\text{all positive}} \qquad \tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{4\pi}{9} = \sqrt{3}$$

Notice that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)((x+y+z)^2 - 3(xy+yz+zx))$

Therefore

$$\begin{aligned}\tan^6\left(\frac{\pi}{9}\right) + \tan^6\left(\frac{2\pi}{9}\right) + \tan^6\left(\frac{4\pi}{9}\right) &= 33(33^2 - 3\cdot 27) + 3\cdot 3 \\ &= 33\,273\end{aligned}$$

Question (1990 STEP III Q4)
Given that $\sin\beta \neq 0$, sum the series

$$\cos\alpha + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+2r\beta) + \cdots + \cos(\alpha+2n\beta)$$

and

$$\cos\alpha + \binom{n}{1}\cos(\alpha+2\beta) + \cdots + \binom{n}{r}\cos(\alpha+2r\beta) + \cdots + \cos(\alpha+2n\beta).$$

Given that $\sin\theta \neq 0$, prove that

$$1+\cos\theta\sec\theta+\cos 2\theta\sec^2\theta+\cdots+\cos r\theta\sec^r\theta+\cdots+\cos n\theta\sec^n\theta = \frac{\sin(n+1)\theta\sec^n\theta}{\sin\theta}.$$

$$\begin{aligned}\sum_{r=0}^{n}\cos(\alpha+2r\beta) &= \sum_{r=0}^{n}\operatorname{Re}\left(\exp(i(\alpha+2r\beta))\right) \\ &= \operatorname{Re}\left(\sum_{r=0}^{n}\exp(i(\alpha+2r\beta))\right) \\ &= \operatorname{Re}\left(e^{i\alpha}\sum_{r=0}^{n}(e^{i2\beta})^r\right) \\ &= \operatorname{Re}\left(e^{i\alpha}\frac{e^{2(n+1)\beta i}-1}{e^{2\beta i}-1}\right) \\ &= \operatorname{Re}\left(e^{i\alpha}\frac{e^{(n+1)\beta i}(e^{(n+1)\beta i}-e^{-(n+1)\beta i})}{e^{\beta i}(e^{\beta i}-e^{-\beta i})}\right) \\ &= \operatorname{Re}\left(\frac{e^{i\alpha}e^{(n+1)\beta i}}{e^{\beta i}}\frac{\sin(n+1)\beta}{\sin\beta}\right) \\ &= \operatorname{Re}\left(e^{i(\alpha+n\beta)}\frac{\sin(n+1)\beta}{\sin\beta}\right) \\ &= \frac{\cos(\alpha+n\beta)\sin(n+1)\beta}{\sin\beta}\end{aligned}$$

$$\begin{aligned}
\sum_{r=0}^{n} \binom{n}{r} \cos(\alpha + 2r\beta) &= \sum_{r=0}^{n} \text{Re} \left(\binom{n}{r} \exp(i(\alpha + 2r\beta)) \right) \\
&= \text{Re} \left(\sum_{r=0}^{n} \binom{n}{r} \exp(i(\alpha + 2r\beta)) \right) \\
&= \text{Re} \left(e^{i\alpha}(e^{2\beta i} + 1)^n \right) \\
&= \text{Re} \left(e^{i\alpha} e^{n\beta i}(e^{\beta i} + e^{-\beta i})^n \right) \\
&= \text{Re} \left(e^{i\alpha} e^{n\beta i} 2^n \cos^n \beta \right) \\
&= 2^n \cos(\alpha + n\beta) \cos^n \beta
\end{aligned}$$

$$\begin{aligned}
\sum_{r=0}^{n} \cos r\theta \sec^r \theta &= \sum_{r=0}^{n} \text{Re}(e^{ir\theta}) \sec^r \theta \\
&= \text{Re} \left(\sum_{r=0}^{n} e^{ir\theta} \sec^r \theta \right) \\
&= \text{Re} \left(\frac{e^{i(n+1)\theta} \sec^{n+1} \theta - 1}{e^{i\theta} \sec \theta - 1} \right) \\
&= \text{Re} \left(\frac{e^{i(n+1)\theta} \sec^{n} \theta - \cos \theta}{e^{i\theta} - \cos \theta} \right) \\
&= \text{Re} \left(\frac{e^{i(n+1)\theta} \sec^{n} \theta - \cos \theta}{i \sin \theta} \right) \\
&= \frac{1}{\sin \theta} \text{Im} \left(e^{i(n+1)\theta} \sec^{n} \theta - \cos \theta \right) \\
&= \frac{\sin(n+1)\theta \sec^n \theta}{\sin \theta}
\end{aligned}$$

Question (1991 STEP I Q3)

A path is made up in the Argand diagram of a series of straight line segments P_1P_2, P_2P_3, $P_3P_4, \dots$ such that each segment is d times as long as the previous one, ($d \neq 1$), and the angle between one segment and the next is always θ (where the segments are directed from P_j towards P_{j+1}, and all angles are measured in the anticlockwise direction). If P_j represents the complex number z_j, express

$$\frac{z_{n+1} - z_n}{z_n - z_{n-1}}$$

as a complex number (for each $n \geqslant 2$), briefly justifying your answer. If $z_1 = 0$ and $z_2 = 1$, obtain an expression for z_{n+1} when $n \geqslant 2$. By considering its imaginary part, or otherwise, show that if $\theta = \frac{1}{3}\pi$ and $d = 2$, then the path crosses the real axis infinitely often.

$$\left|\frac{z_{n+1} - z_n}{z_n - z_{n-1}}\right| = d$$

$$\begin{aligned}\arg\left(\frac{z_{n+1} - z_n}{z_n - z_{n-1}}\right) &= \arg(z_{n+1} - z_n) - \arg(z_n - z_{n-1})\\ &= \theta\\ \Rightarrow \qquad \frac{z_{n+1} - z_n}{z_n - z_{n-1}} &= de^{i\theta}\end{aligned}$$

$$\begin{aligned} z_1 &= 0\\ z_2 &= 1\\ \frac{z_3 - z_2}{z_2 - z_1} &= de^{i\theta}\\ \Rightarrow \qquad z_3 &= de^{i\theta} + 1\\ \frac{z_4 - z_3}{z_3 - z_2} &= de^{i\theta}\\ \Rightarrow \qquad z_4 &= (de^{i\theta})^2 + de^{i\theta} + 1\\ \Rightarrow \qquad z_{n+1} &= \frac{(de^{i\theta})^n - 1}{de^{i\theta} - 1}\end{aligned}$$

If $d = 2, \theta = \frac{1}{3}\pi$, then, $2e^{i\frac{1}{3}\pi} = 1 + \sqrt{3}i$

$$\begin{aligned}\text{Im}(z_{n+1})) &= \text{Im}\left(\frac{(2e^{i\frac{1}{3}\pi})^n - 1}{2e^{i\frac{1}{3}\pi} - 1}\right)\\ &= \text{Im}\left(\frac{(2e^{i\frac{1}{3}\pi})^n - 1}{\sqrt{3}i}\right)\\ &= -\frac{1}{\sqrt{3}}\text{Re}\left(2^n e^{i\frac{n}{3}\pi}\right) + \frac{1}{\sqrt{3}}\end{aligned}$$

Which clearly changes sign infinitely many times, ie crosses the origin infinitely many times.

Question (1992 STEP III Q8)

Show that

$$\sin(2n+1)\theta = \sin^{2n+1}\theta \sum_{r=0}^{n}(-1)^{n-r}\binom{2n+1}{2r}\cot^{2r}\theta,$$

where n is a positive integer. Deduce that the equation

$$\sum_{r=0}^{n}(-1)^{r}\binom{2n+1}{2r}x^{r} = 0$$

has roots $\cot^2(k\pi/(2n+1))$ for $k = 1, 2, \ldots, n$.

Show that

(i) $\displaystyle\sum_{k=1}^{n}\cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)}{3},$

(ii) $\displaystyle\sum_{k=1}^{n}\tan^2\left(\frac{k\pi}{2n+1}\right) = n(2n+1),$

(iii) $\displaystyle\sum_{k=1}^{n}\operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n(n+1)}{3}.$

Question (1995 STEP I Q4)

By applying de Moivre's theorem to $\cos 5\theta + \mathrm{i}\sin 5\theta$, expanding the result using the binomial theorem, and then equating imaginary parts, show that

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

Use this identity to evaluate $\cos^2\frac{1}{5}\pi$, and deduce that $\cos\frac{1}{5}\pi = \frac{1}{4}(1+\sqrt{5})$.

$$\begin{aligned}
& & (\cos\theta + i\sin\theta)^n &= \cos n\theta + i\sin n\theta \\
n = 5: & & \cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\
\text{Im}: & & \sin 5\theta &= \binom{5}{1}\cos^4\theta\sin\theta + \binom{5}{3}\cos^2\theta(-\sin^3\theta) + \binom{5}{5}\sin^5\theta \\
& & &= \sin\theta(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta) \\
& & &= \sin\theta(5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2) \\
& & &= \sin\theta((5+10+1)\cos^4\theta + (-10-2)\cos^2\theta + 1) \\
& & &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1)
\end{aligned}$$

Suppose $\theta = \frac{\pi}{5}$, then $\sin 5\theta = 0, \sin\theta \neq 0$, therefore if $c = \cos\theta$ we must have

$$\begin{aligned} 0 &= 16c^4 - 12c^2 + 1 \\ \Rightarrow \quad c^2 &= \frac{3 \pm \sqrt{5}}{8} \\ &= \frac{6 \pm 2\sqrt{5}}{16} \\ &= \frac{(1 \pm \sqrt{5})^2}{16} \\ \Rightarrow \quad c &= \pm \frac{1 \pm \sqrt{5}}{4} \end{aligned}$$

Since $c > 0$ we either have $\cos \frac{1}{5}\pi = \frac{1+\sqrt{5}}{4}$ or $\cos \frac{1}{5}\pi = \frac{\sqrt{5}-1}{4}$, however $\sqrt{5} - 1 < 1.5$ and so $\frac{\sqrt{5}-1}{4} < \frac{1}{2} = \cos \frac{1}{3}\pi$ we must have $\cos \frac{1}{5}\pi = \frac{1+\sqrt{5}}{4}$

Question (1995 STEP II Q6)

If u and v are the two roots of $z^2 + az + b = 0$, show that $a = -u - v$ and $b = uv$.

Let $\alpha = \cos(2\pi/7) + \mathrm{i}\sin(2\pi/7)$. Show that α is a root of $z^6 - 1 = 0$ and express the roots in terms of α. The number $\alpha + \alpha^2 + \alpha^4$ is a root of a quadratic equation

$$z^2 + Az + B = 0$$

where A and B are real. By guessing the other root, or otherwise, find the numerical values of A and B.

Show that

$$\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} = -\frac{1}{2},$$

and evaluate

$$\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{8\pi}{7},$$

making it clear how you determine the sign of your answer.

$$\begin{aligned} 0 &= z^2 + az + b \\ &= (z - u)(z - v) \\ &= z^2 - (u + v)z + uv \end{aligned}$$

Therefore by comparing coefficients, $a = -u - v$ and $b = uv$.

Suppose $\alpha = \cos(2\pi/7) + i\sin(2\pi/7)$, then by De Moivre, $\alpha^7 = \cos(2\pi) + i\sin(2\pi) = 1$, ie $\alpha^7 - 1 = 0$.

Notice that $(\alpha + \alpha^2 + \alpha^4) + (\alpha^3 + \alpha^5 + \alpha^6) = -1$ and

$$\begin{aligned} P &= (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6) \\ &= \alpha^4 + \alpha^6 + \alpha^7 + \alpha^5 + \alpha^7 + \alpha^8 + \alpha^7 + \alpha^9 + \alpha^{10} \\ &= 3 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 \\ &= 2 \end{aligned}$$

Therefore it is a root of $x^2 + x + 2 = 0 \Rightarrow x = \frac{-1 \pm i\sqrt{7}}{2}$

Therefore $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = \mathrm{Re}(\alpha + \alpha^2 + \alpha^4) = -\frac{1}{2}$

And $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \mathrm{Im}(\alpha + \alpha^2 + \alpha^4) = \pm\frac{\sqrt{7}}{2}$ since it is positive it is $\frac{\sqrt{7}}{2}$

Question (1996 STEP III Q5)

Show, using de Moivre's theorem, or otherwise, that

$$\tan 7\theta = \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1},$$

where $t = \tan\theta$.

(i) By considering the equation $\tan 7\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{7}\right), \ \tan^2\left(\frac{2\pi}{7}\right) \ \text{and} \ \tan^2\left(\frac{3\pi}{7}\right)$$

and deduce the value of

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

(ii) Find, without using a calculator, the value of

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right).$$

None

Question (1997 STEP III Q3)

By considering the solutions of the equation $z^n - 1 = 0$, or otherwise, show that

$$(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1}) = 1 + z + z^2 + \dots + z^{n-1},$$

where z is any complex number and $\omega = e^{2\pi i/n}$. Let $A_1, A_2, A_3, \dots, A_n$ be points equally spaced around a circle of radius r centred at O (so that they are the vertices of a regular n-sided polygon). Show that

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \dots + \overrightarrow{OA_n} = \mathbf{0}.$$

Deduce, or prove otherwise, that

$$\sum_{k=1}^{n} |A_1A_k|^2 = 2r^2n.$$

Question (2000 STEP II Q4)

Prove that

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) = \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$$

and that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

By considering $(5-\mathrm{i})^2(1+\mathrm{i})$, or otherwise, prove that

$$\arctan\left(\frac{7}{17}\right) + 2\arctan\left(\frac{1}{5}\right) = \frac{\pi}{4}\,.$$

Prove also that

$$3\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right) = \frac{\pi}{4}\,.$$

[Note that $\arctan\theta$ is another notation for $\tan^{-1}\theta$.]

Question (2000 STEP III Q3)

Given that $\alpha = \mathrm{e}^{\mathrm{i}\pi/3}$, prove that $1+\alpha^2 = \alpha$.

A triangle in the Argand plane has vertices A, B, and C represented by the complex numbers p, $q\alpha^2$ and $-r\alpha$ respectively, where p, q and r are positive real numbers. Sketch the triangle ABC.

Three equilateral triangles ABL, BCM and CAN (each lettered clockwise) are erected on sides AB, BC and CA respectively. Show that the complex number representing N is $(1-\alpha)p - \alpha^2 r$ and find similar expressions for the complex numbers representing L and M.

Show that lines LC, MA and NB all meet at the origin, and that these three line segments have the common length $p+q+r$.

Question (2005 STEP III Q6)

In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh(3y) \quad \text{and} \quad \operatorname{arcosh} y = \ln(y + \sqrt{y^2 - 1}).$$

[**Note**: arcosh$y$ is another notation for $\cosh^{-1} y$] Show that the equation $x^3 - 3a^2 x = 2a^3 \cosh T$ is satisfied by $2a\cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geq b^3 > 0$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \dfrac{b}{u}$, where $u = (c + \sqrt{c^2 - b^3})^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation

$$x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0\,,$$

and find these roots in terms of u, b and ω, where $\omega = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.

Question (2009 STEP III Q6)

Show that $\left|e^{\mathrm{i}\beta} - e^{\mathrm{i}\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$\left|e^{\mathrm{i}\alpha} - e^{\mathrm{i}\beta}\right| \left|e^{\mathrm{i}\gamma} - e^{\mathrm{i}\delta}\right| + \left|e^{\mathrm{i}\beta} - e^{\mathrm{i}\gamma}\right| \left|e^{\mathrm{i}\alpha} - e^{\mathrm{i}\delta}\right| = \left|e^{\mathrm{i}\alpha} - e^{\mathrm{i}\gamma}\right| \left|e^{\mathrm{i}\beta} - e^{\mathrm{i}\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$. Interpret this result as a theorem about cyclic quadrilaterals.

Question (2010 STEP III Q3)

For any given positive integer n, a number a (which may be complex) is said to be a *primitive nth root of unity* if $a^n = 1$ and there is no integer m such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity. Let $\mathrm{C}_n(x)$ be the polynomial such that the roots of the equation $\mathrm{C}_n(x) = 0$ are the primitive nth roots of unity, the coefficient of the highest power of x is one and the equation has no repeated roots. Show that $\mathrm{C}_4(x) = x^2 + 1$.

(i) Find $\mathrm{C}_1(x)$, $\mathrm{C}_2(x)$, $\mathrm{C}_3(x)$, $\mathrm{C}_5(x)$ and $\mathrm{C}_6(x)$, giving your answers as unfactorised polynomials.

(ii) Find the value of n for which $\mathrm{C}_n(x) = x^4 + 1$.

(iii) Given that p is prime, find an expression for $\mathrm{C}_p(x)$, giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers q, r and s such that $\mathrm{C}_q(x) \equiv \mathrm{C}_r(x)\mathrm{C}_s(x)$.

The primitive 4th roots of unity are i and $-i$. (Since the other two roots of x^4-1 are also roots of x^2-1

$C_4(x) = (x-i)(x+i) = x^2+1$ as required.

(i)

$$\begin{aligned} C_1(x) &= x-1 \\ C_2(x) &= x+1 \\ C_3(x) &= x^2+x+1 \\ C_5(x) &= x^4+x^3+x^2+x+1 \\ C_6(x) &= x^2-x+1 \end{aligned}$$

(ii) Since $(x^4+1)(x^4-1) = x^8-1$ we must have $n \mid 8$. But $n \neq 1, 2, 4$ so $n = 8$.

(iii) $C_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$

(iv) Suppose $C_q(x) \equiv C_r(x)C_s(x)$, then if ω is a primitive qth root of unity we must $C_q(\omega) = 0$, but that means that one of $C_r(\omega)$, $C_s(\omega)$ is 0. But that's only possible if r or $s = q$. If this were the case, then what would the other value be? There are no possible values, hence it's not possible.

Question (2011 STEP III Q3)

Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \qquad\qquad (p \neq 0,\ q \neq 0)$$

can be written in the form

$$a(x-\alpha)^3 + b(x-\beta)^3\,,$$

where α and β are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and a and b are constants which you should express in terms of α and β. Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2(2-2^{\frac{1}{3}})}{1-2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of ω, where $\omega = e^{2\pi i/3}$.

Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant r.

Question (2013 STEP III Q4)

Show that $(z - e^{i\theta})(z - e^{-i\theta}) = z^2 - 2z\cos\theta + 1$. Write down the $(2n)$th roots of -1 in the form $e^{i\theta}$, where $-\pi < \theta \le \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right) + 1 \right).$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = i$ show that, when n is even,

$$\cos\left(\tfrac{\pi}{2n}\right)\cos\left(\tfrac{3\pi}{2n}\right)\cos\left(\tfrac{5\pi}{2n}\right)\cdots\cos\left(\tfrac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\tfrac{\pi}{2n}\right)\cos^2\left(\tfrac{3\pi}{2n}\right)\cos^2\left(\tfrac{5\pi}{2n}\right)\cdots\cos^2\left(\tfrac{(n-2)\pi}{2n}\right) = n2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1+z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

$$\begin{aligned}(z - e^{i\theta})(z - e^{-i\theta}) &= z^2 - (e^{i\theta} + e^{-i\theta})z + 1 \\ &= z^2 - 2\cos\theta z + 1\end{aligned}$$

The $2n$th roots of -1 are $e^{\frac{i(2k+1)\pi}{2n}}, k \in \{-n, \cdots, n-1\}$ or $e^{\frac{ik\pi}{2n}}, k \in \{-2n+1, -2n+3, \cdots, 2n-3, 2n-1\}$

$$\begin{aligned} z^{2n} + 1 &= (z - e^{-i(2n-1)/2n}) \cdot (z - e^{-i(2n-3)/2n}) \cdots (z - e^{i(2n-3)/2n}) \cdot (z - e^{i(2n-1)/2n}) \\ &= \prod_{k=1}^{n} \left(z - e^{i\frac{2k-1}{2n}\pi}\right)\left(z - e^{-i\frac{2k-1}{2n}\pi}\right) \\ &= \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1 \right) \end{aligned}$$

(i)

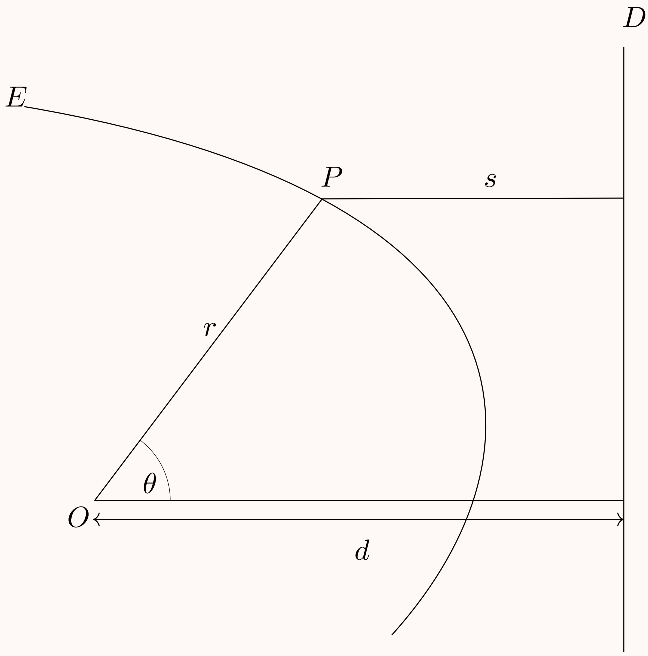
$$\begin{aligned} i^{2n} + 1 &= \prod_{k=1}^{n}\left(i^2 - 2i\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1\right) \\ \Rightarrow \quad (-1)^n + 1 &= (-1)^n 2^n i^n \prod_{k=1}^{n} \cos\left(\frac{(2k-1)\pi}{2n}\right) \\ \Rightarrow \quad \prod_{k=1}^{n}\cos\left(\frac{(2k-1)\pi}{2n}\right) &= 2^{1-n}(-1)^{n/2} \qquad (\text{if } n \equiv 0 \pmod 2) \end{aligned}$$

(ii) When n is odd, we notice that two of the roots are i and $-i$, if we exclude those, (ie by factoring out $z^2 + 1$, we see that

$$
\begin{aligned}
1 - z^2 + z^4 - \cdots + z^{2n-2} &= \prod_{k=1, 2k-1 \neq n}^{n} \left(z^2 - 2z\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1 \right) \\
&= \prod_{k=1}^{(n-1)/2} \left(z^2 - 2z\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1 \right) \prod_{k=(n+1)/2}^{n} \Big(z^2 - 2z\cos\Big((2k \\
&= \prod_{k=1}^{(n-1)/2} \left(z^2 - 2z\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1 \right) \prod_{k=1}^{(n-1)/2} \Big(z^2 + 2z\cos\Big(\frac{(2k -}{2n} \\
\Rightarrow \quad 1 - i^2 + i^4 + \cdots + i^{2n-2} &= \prod_{k=1}^{(n-1)/2} \left(2\cos\left(\frac{(2k-1)\pi}{2n}\right) \right) \prod_{k=1}^{(n-1)/2} \left(2\cos\left(\frac{(2k-1)\pi}{2n}\right) \right) \\
\Rightarrow \quad n &= 2^{n-1} \prod_{k=1}^{(n-1)/2} \cos^2\left(\frac{(2k-1)\pi}{2n}\right) \\
\Rightarrow \quad \prod_{k=1}^{(n-1)/2} \cos^2\left(\frac{(2k-1)\pi}{2n}\right) &= n2^{1-n}
\end{aligned}
$$

Question (2013 STEP III Q8)

Evaluate $\sum_{r=0}^{n-1} e^{2i(\alpha+r\pi/n)}$ where α is a fixed angle and $n \geq 2$. The fixed point O is a distance d from a fixed line D. For any point P, let s be the distance from P to D and let r be the distance from P to O. Write down an expression for s in terms of d, r and the angle θ, where θ is as shown in the diagram below.



The curve E shown in the diagram is such that, for any point P on E, the relation $r = ks$ holds, where k is a fixed number with $0 < k < 1$. Each of the n lines $L_1, \ldots, L_n$ passes through O and the angle between adjacent lines is $\frac{\pi}{n}$. The line L_j ($j = 1, \ldots, n$) intersects E in two points forming a chord of length l_j. Show that, for $n \geq 2$,

$$\sum_{j=1}^{n} \frac{1}{l_j} = \frac{(2-k^2)n}{4kd}.$$

$$\begin{aligned}
\sum_{r=0}^{n-1} e^{2i(\alpha+r\pi/n)} &= e^{2i\alpha} \sum_{r=0}^{n-1} \left(e^{2i\pi/n}\right)^r \\
&= e^{2i\alpha} \frac{1-\left(e^{2i\pi/n}\right)^n}{1-e^{2i\pi/n}} \\
&= 0
\end{aligned}$$

$d = s + r\cos\theta$ ie $s = d - r\cos\theta$

Therefore $d = \frac{r}{k} + r\cos\theta \Rightarrow r = \frac{kd}{1+k\cos\theta}$. The l_j will come from $r(\alpha+\frac{j\pi}{n}) + r(\alpha+\pi+\frac{j\pi}{n})$

$$l_j = r(\alpha + \frac{(j-1)\pi}{n}) + r(\alpha + \pi + \frac{(j-1)\pi}{n})$$

$$
\begin{aligned}
&= \frac{kd}{1+k\cos\left(\alpha+\frac{(j-1)\pi}{n}\right)} + \frac{kd}{1+k\cos\left(\alpha+\pi+\frac{(j-1)\pi}{n}\right)} \\
&= \frac{kd}{1+k\cos\left(\alpha+\frac{(j-1)\pi}{n}\right)} + \frac{kd}{1-k\cos\left(\alpha+\frac{(j-1)\pi}{n}\right)} \\
&= \frac{2kd}{1-k^2\cos^2\left(\alpha+\frac{(j-1)\pi}{n}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad \sum_{j=1}^{n}\frac{1}{l_j} &= \sum_{j=0}^{n-1}\frac{1-k^2\cos^2\left(\alpha+\frac{j\pi}{n}\right)}{2kd} \\
&= \frac{n}{2kd} - \frac{k^2}{2kd}\sum_{j=0}^{n-1}\cos^2\left(\alpha+\frac{j\pi}{n}\right) \\
&= \frac{n}{2kd} - \frac{k^2}{2kd}\sum_{j=0}^{n-1}\frac{1+\cos\left(2\alpha+\frac{2j\pi}{n}\right)}{2} \\
&= \frac{n}{2kd} - \frac{nk^2}{2kd} - \frac{k^2}{4kd}\sum_{j=0}^{n-1}\cos\left(2\alpha+\frac{2j\pi}{n}\right) \\
&= \frac{n}{2kd} - \frac{nk^2}{2kd} - \frac{k^2}{4kd}\underbrace{\operatorname{Re}\left(\sum_{j=0}^{n-1}e^{2i\left(\alpha+\frac{j\pi}{n}\right)}\right)}_{=0} \\
&= \frac{n}{2kd} - \frac{nk^2}{4kd} \\
&= \frac{n(2-k^2)}{4kd}
\end{aligned}
$$

Question (2015 STEP III Q4)**(i)** If a, b and c are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

has at least one real root.

(ii) Let

$$S_1 = z_1 + z_2 + z_3, \qquad S_2 = z_1^2 + z_2^2 + z_3^2, \qquad S_3 = z_1^3 + z_2^3 + z_3^3,$$

where z_1, z_2 and z_3 are the roots of the equation $(*)$. Express a and b in terms of S_1 and S_2, and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3.$$

(iii) The six real numbers r_k and θ_k $(k = 1,\ 2,\ 3)$, where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^{3} r_k \sin(\theta_k) = 0\,, \qquad \sum_{k=1}^{3} r_k^2 \sin(2\theta_k) = 0\,, \qquad \sum_{k=1}^{3} r_k^3 \sin(3\theta_k) = 0\,.$$

Show that $\theta_k = 0$ for at least one value of k. Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.

(i) Let $z \in \mathbb{R}$ and let $z \to \pm\infty$ then $z^3 + az^2 + bz + c$ changes sign, therefore somewhere it must have a real root.

(ii)

$$\begin{aligned} z^3 + az^2 + bz + c &= (z - z_1)(z - z_2)(z - z_3) \\ &= z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_2z_3 + z_3z_1)z - (z_1z_2z_3) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad S_1 &= z_1 + z_2 + z_3 \\ &= -a \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad S_2 &= z_1^2 + z_2^2 + z_3^2 \\ &= (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1) \\ &= a^2 - 2b \\ \Rightarrow \qquad a &= -S_1 \\ b &= \frac{1}{2}\left(S_1^2 - S_2\right) \end{aligned}$$

$$\begin{aligned} 0 &= z_i^3 + az_i^2 + bz_i + c \\ \Rightarrow \qquad 0 &= S_3 + aS_2 + bS_1 + 3c \\ &= S_3 - S_1S_2 + \frac{1}{2}\left(S_1^2 - S_2\right)S_1 + 3c \end{aligned}$$

$$\Rightarrow \qquad 0 = 2S_3 - 3S_1S_2 + S_1^3 + 6c$$

(iii) Let $z_k = r_k e^{i\theta_k}$, then we have $\text{Im}(S_k) = 0$ and so the polynomial with roots z_k has real coefficients, and therefore at least one root is real. This root will have $\theta_k = 0$. Moreover, since if w is a root of a real polynomial w is also a root, and therefore if $\theta_1 = 0$, we must have that z_2 and z_3 are complex conjugate, ie $\theta_2 = -\theta_3$

Question (2016 STEP III Q7)

Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$, where n is a positive integer. Show that, for any complex number z,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1\,.$$

The points X_0, X_1, ..., X_{n-1} lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

(i) The point P is equidistant from X_0 and X_1. Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1\,,$$

where $|PX_k|$ denotes the distance from P to X_k.

Give the corresponding result when n is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n\,.$$

None

Question (2017 STEP III Q2)

The transformation R in the complex plane is a rotation (anticlockwise) by an angle θ about the point represented by the complex number a. The transformation S in the complex plane is a rotation (anticlockwise) by an angle ϕ about the point represented by the complex number b.

(i) The point P is represented by the complex number z. Show that the image of P under R is represented by

$$e^{i\theta} z + a(1 - e^{i\theta}) .$$

(ii) Show that the transformation SR (equivalent to R followed by S) is a rotation about the point represented by c, where

$$c \sin \tfrac{1}{2}(\theta + \phi) = a\, e^{i\phi/2} \sin \tfrac{1}{2}\theta + b\, e^{-i\theta/2} \sin \tfrac{1}{2}\phi ,$$

provided $\theta + \phi \neq 2n\pi$ for any integer n.

What is the transformation SR if $\theta + \phi = 2\pi$?

(iii) Under what circumstances is $RS = SR$?

(i) We can map $a \mapsto 0$, rotate the whole plane, then shift the plane back to a, so

$$z \mapsto (z - a) \mapsto e^{i\theta}(z - a) \mapsto a + e^{i\theta}(z - a) = e^{i\theta} z + a(1 - e^{i\theta})$$

(ii) $z \mapsto e^{i\theta} z + a(1 - e^{i\theta}) \mapsto e^{i\phi} \left(e^{i\theta} z + a(1 - e^{i\theta}) \right) + b(1 - e^{i\phi})$

$$e^{i\phi} \left(e^{i\theta} z + a(1 - e^{i\theta}) \right) + b(1 - e^{i\phi}) = e^{i(\phi+\theta)} z + a e^{i\phi} - a e^{i(\theta+\phi)} + b(1 - e^{i\phi})$$

Therefore this is rotation by angle $\phi + \theta$ and about

$$\begin{aligned}
\frac{a e^{i\phi} - a e^{i(\theta+\phi)} + b(1 - e^{i\phi})}{1 - e^{i(\phi+\theta)}} &= \frac{e^{-i\frac{(\phi+\theta)}{2}} \left(a e^{i\phi} - a e^{i(\theta+\phi)} + b(1 - e^{i\phi}) \right)}{e^{-i\frac{(\phi+\theta)}{2}} - e^{i\frac{(\phi+\theta)}{2}}} \\
&= \frac{\left(a e^{i\frac{\phi-\theta}{2}} - a e^{i\frac{(\theta+\phi)}{2}} + b(e^{-i\frac{(\phi+\theta)}{2}} - e^{i\frac{(\phi-\theta)}{2}}) \right)}{e^{-i\frac{(\phi+\theta)}{2}} - e^{i\frac{(\phi+\theta)}{2}}} \\
&= \frac{a e^{i\frac{\phi}{2}} 2i \sin(\frac{\theta}{2}) + b e^{-i\frac{\theta}{2}} 2i \sin\frac{\phi}{2}}{2i \sin(\frac{\phi+\theta}{2})}
\end{aligned}$$

as required.

If $\phi + \theta = 2\pi$, then $z \mapsto z + (b - a)(1 - e^{i\phi})$ which is a translation.

(iii) If $\phi + \theta \neq 2\pi$ then $RS = ST$ if

$$a\,\mathrm{e}^{\mathrm{i}\phi/2}\sin\tfrac{1}{2}\theta + b\,\mathrm{e}^{-\mathrm{i}\theta/2}\sin\tfrac{1}{2}\phi = b\,\mathrm{e}^{\mathrm{i}\theta/2}\sin\tfrac{1}{2}\phi + a\,\mathrm{e}^{-\mathrm{i}\phi/2}\sin\tfrac{1}{2}\theta$$

$$a\,(\mathrm{e}^{\mathrm{i}\phi/2} - \mathrm{e}^{-\mathrm{i}\phi/2})\sin\tfrac{1}{2}\theta + b\,(\mathrm{e}^{-\mathrm{i}\theta/2} - \mathrm{e}^{+\mathrm{i}\theta/2})\sin\tfrac{1}{2}\phi = 0$$

$$a\sin\frac{\phi}{2}\sin\frac{\theta}{2} - b\sin\frac{\theta}{2}\sin\frac{\phi}{2} = 0$$

$$\Leftrightarrow \quad a = b \text{ or } \sin\frac{\theta}{2} = 0 \text{ or } \sin\frac{\phi}{2} = 0$$

$$\Leftrightarrow \quad a = b \text{ or } \theta = 0 \text{ or } \phi = 0$$

If $\phi + \theta \neq 2\pi$ then $RS = ST$ if $b = a$ or $e^{i\phi} = e^{i\theta}$ ie rotation through the same angle.

Question (2018 STEP III Q7)**(i)** Use De Moivre's theorem to show that, if $\sin\theta \neq 0$, then

$$\frac{(\cot\theta + \mathrm{i})^{2n+1} - (\cot\theta - \mathrm{i})^{2n+1}}{2\mathrm{i}} = \frac{\sin{(2n+1)\,\theta}}{\sin^{2n+1}\theta}\,,$$

for any positive integer n. Deduce that the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \cdots + (-1)^n = 0$$

are

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right)$$

where $m = 1,\, 2,\, \ldots\,,\, n\,.$

(ii) Hence show that

$$\sum_{m=1}^{n}\cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n\,(2n-1)}{3}.$$

(iii) Given that $0 < \sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{1}{2}\pi$, show that

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta.$$

Hence show that

$$\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}\,.$$

(i)

$$\frac{(\cot\theta + i)^{2n+1} - (\cot\theta - i)^{2n+1}}{2i} = \frac{1}{\sin^{2n+1}\theta}\,\frac{(\cos\theta + i\sin\theta)^{2n+1} - (\cos\theta - i\sin\theta)^{2n+1}}{2i}$$

$$= \frac{1}{\sin^{2n+1}\theta}\frac{e^{i(2n+1)\theta} - e^{-i(2n+1)\theta}}{2i}$$
$$= \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$

Notice that:

$$\begin{aligned}(\cot\theta + i)^{2n+1} - (\cot\theta - i)^{2n+1} &= \sum_{k=0}^{2n+1}\binom{2n+1}{k}(i)^k \cdot \cot^{2n+1-k}\theta - \sum_{k=0}^{2n+1}\binom{2n+1}{k}(-i)^k \cdot \cot^{2n+1-}\\
&= \sum_{k=0}^{2n+1}\binom{2n+1}{k}\left(i^k - (-i)^k\right)\cdot\cot^{2n+1-k}\theta\\
&= \sum_{l=0}^{n}\binom{2n+1}{2l+1}\left(i^{2l+1} - (-i)^{2l+1}\right)\cdot\cot^{2n+1-(2l+1)}\theta\\
&= \sum_{l=0}^{n}\binom{2n+1}{2l+1}2i\cdot\cot^{2(n-l)}\theta\\
&= 2i\sum_{l=0}^{n}\binom{2n+1}{2l+1}\cot^{2(n-l)}\theta\end{aligned}$$

Therefore if θ satisfies $\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = 0$ then $z = \cot^2\theta$ satisfies the equation. But $\theta = \frac{m\pi}{2n+1}, m = 1, 2, \ldots, n$ are n distinct all the roots must be $\cot^2\frac{m\pi}{2n+1}$.

(ii) Notice that the sum of the roots will be $\frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)\cdot 2n\cdot(2n-1)}{3!\cdot(2n+1)} = \frac{n\cdot(2n-1)}{3}$ and so

$$\sum_{m=1}^{n}\cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n\,(2n-1)}{3}.$$

(iii) For $0 < \theta < \frac{1}{2}\pi$,

$$\begin{aligned}&& 0 < \sin\theta < \theta < \tan\theta\\
&\Leftrightarrow & 0 < \cot\theta < \frac{1}{\theta} < \frac{1}{\sin\theta}\\
&\Leftrightarrow & 0 < \cot^2\theta < \frac{1}{\theta^2} <^2\theta = 1 + \cot^2\theta\end{aligned}$$

Therefore

$$\sum_{n=1}^{N}\cot^2\frac{n\pi}{2N+1} < \sum_{n=1}^{N}\frac{(2N+1)^2}{n^2\pi^2} < N + \sum_{n=1}^{N}\cot^2\frac{n\pi}{2N+1}$$

$$\Rightarrow \quad \frac{1}{(2N+1)^2}\frac{N(2N-1)}{3} < \sum_{n=1}^{N}\frac{1}{n^2\pi^2} < \frac{1}{(2N+1)^2}\left(\frac{N(2N-1)}{3}+1\right)$$

$$\Rightarrow \quad \lim_{N\to\infty}\frac{1}{(2N+1)^2}\frac{N(2N-1)}{3} < \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2\pi^2} < \lim_{N\to\infty}\frac{1}{(2N+1)^2}\left(\frac{N(2N-1)}{3}+1\right)$$

$$\Rightarrow \quad \frac{1}{6} \leq \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2\pi^2} \leq \frac{1}{6}$$

$$\Rightarrow \quad \sum_{n=1}^{N}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Question (1987 STEP I Q7)

Sum each of the series

$$\sin\left(\frac{2\pi}{23}\right)+\sin\left(\frac{6\pi}{23}\right)+\sin\left(\frac{10\pi}{23}\right)+\cdots+\sin\left(\frac{38\pi}{23}\right)+\sin\left(\frac{42\pi}{23}\right)$$

and

$$\sin\left(\frac{2\pi}{23}\right)-\sin\left(\frac{6\pi}{23}\right)+\sin\left(\frac{10\pi}{23}\right)-\cdots-\sin\left(\frac{38\pi}{23}\right)+\sin\left(\frac{42\pi}{23}\right),$$

giving each answer in terms of the tangent of a single angle.

[No credit will be given for a numerical answer obtained purely by use of a calculator.]

$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. Also let $z = e^{\frac{2\pi i}{23}}$

$$\begin{aligned}
\sum_{k=0}^{10}\sin\left(\frac{(4k+2)\pi}{23}\right) &= \sum_{k=0}^{10}\text{Im}\left(\exp\left(\frac{(4k+2)\pi i}{23}\right)\right)\\
&= \text{Im}\left(\sum_{k=0}^{10}\exp\left(\frac{(4k+2)\pi i}{23}\right)\right)\\
&= \text{Im}\left(e^{\frac{2\pi i}{23}}\sum_{k=0}^{10}z^{2k}\right)\\
&= \text{Im}\left(z\left(\frac{z^{22}-1}{z^2-1}\right)\right)\\
&= \text{Im}\left(z\left(\frac{z^{11}(z^{11}-z^{-11})}{z(z-z^{-1})}\right)\right)\\
&= \text{Im}\left(\frac{z^{11}2i\sin\frac{22\pi}{23}}{2i\sin\frac{2\pi}{23}}\right) && \textit{enumi}\\
&= \frac{\sin\frac{22\pi}{23}}{\sin\frac{2\pi}{23}}\text{Im}(z^{11})\\
&= \frac{\sin^2\frac{22\pi}{23}}{\sin\frac{2\pi}{23}}
\end{aligned}$$

$$= \frac{\sin^2 \frac{\pi}{23}}{2 \sin \frac{\pi}{23} \cos \frac{\pi}{23}}$$
$$= \frac{1}{2} \tan \frac{\pi}{23}$$

Similarly,

$$\begin{aligned}
\sum_{k=0}^{10} (-1)^k \sin\left(\frac{(4k+2)\pi}{23}\right) &= \sum_{k=0}^{10} \operatorname{Im}\left((-1)^k \exp\left(\frac{(4k+2)\pi i}{23}\right)\right) \\
&= \operatorname{Im}\left(\sum_{k=0}^{10} (-1)^k \exp\left(\frac{(4k+2)\pi i}{23}\right)\right) \\
&= \operatorname{Im}\left(e^{\frac{2\pi i}{23}} \sum_{k=0}^{10} (-1)^k z^{2k}\right) \\
&= \operatorname{Im}\left(z\left(\frac{z^{22}+1}{z^2+1}\right)\right) \\
&= \operatorname{Im}\left(z\left(\frac{z^{11}(z^{11}+z^{-11})}{z(z+z^{-1})}\right)\right) \\
&= \operatorname{Im}\left(\frac{z^{11} 2\cos\frac{22\pi}{23}}{2\cos\frac{2\pi}{23}}\right) && \textit{enumi} \\
&= \frac{\cos\frac{22\pi}{23}}{\cos\frac{2\pi}{23}} \operatorname{Im}(z^{11}) \\
&= \frac{\cos\frac{22\pi}{23}\sin\frac{22\pi}{23}}{\cos\frac{2\pi}{23}} \\
&= \frac{1}{2}\frac{\sin\frac{44\pi}{23}}{\cos\frac{2\pi}{23}} \\
&= \frac{1}{2}\frac{-\sin\frac{2\pi}{23}}{\cos\frac{2\pi}{23}} \\
&= -\frac{1}{2}\tan\frac{2\pi}{23}
\end{aligned}$$

Question (1987 STEP II Q4)

Explain the geometrical relationship between the points in the Argand diagram represented by the complex numbers z and $z\mathrm{e}^{\mathrm{i}\theta}$. Write down necessary and sufficient conditions that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle taken in anticlockwise order. Show that α, β and γ represent the vertices of an equilateral triangle (taken in any order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Find necessary and sufficient conditions on the complex coefficients a, b and c for the roots of the equation

$$z^3 + az^2 + bz + c = 0$$

to lie at the vertices of an equilateral triangle in the Argand digram.

The point $ze^{i\theta}$ is obtained by rotating the point z about 0 by an angle θ anticlockwise.

The complex numbers α, β and γ will form an equilateral triangle iff the angles between each side are $\frac{\pi}{3}$, ie

$$\begin{cases} \gamma - \beta &= e^{i\frac{\pi}{3}}(\beta - \alpha) \\ \alpha - \gamma &= e^{i\frac{\pi}{3}}(\gamma - \beta) \\ \beta - \alpha &= e^{i\frac{\pi}{3}}(\alpha - \gamma) \end{cases}$$

We don't need all these equations, since the first two are equivalent to the third.

Combining the first two equations, we have

$$\begin{aligned} && \frac{\gamma - \beta}{\beta - \alpha} &= \frac{\alpha - \gamma}{\gamma - \beta} \\ \Leftrightarrow && (\gamma - \beta)^2 &= (\alpha - \gamma)(\beta - \alpha) \\ \Leftrightarrow && \gamma^2 + \beta^2 - 2\gamma\beta &= \alpha\beta - \alpha^2 - \gamma\beta + \gamma\alpha \\ \Leftrightarrow && \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta &= 0 \end{aligned}$$

as required.

If the roots of $z^3+az^2+bz+c = 0$ are α, β, γ then $\alpha+\beta+\gamma = -a$ and $\beta\gamma+\gamma\alpha+\alpha\beta = b$. We also have that $a^2 - 2b = \alpha^2 + \beta^2 + \gamma^2$. Therefore there roots will lie at the vertices of an equilateral triangle iff $a^2 - 3b = 0$

Question (1987 STEP III Q3)**(i)** If $z = x + \mathrm{i}y$, with x, y real, show that

$$|x|\cos\alpha + |y|\sin\alpha \leqslant |z| \leqslant |x| + |y|$$

for all real α.

(ii) By considering $(5 - \mathrm{i})^4(1 + \mathrm{i})$, show that

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Prove similarly that

$$\frac{\pi}{4} = 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{20}\right) + \tan^{-1}\left(\frac{1}{1985}\right).$$

(i) If $z = x + iy$ then $|z|^2 = x^2 + y^2 \leq x^2 + y^2 + 2|x||y| \leq (|x| + |y|)^2$.

The LHS is Cauchy-Schwarz with the vectors $\begin{pmatrix}|x| \\ |y|\end{pmatrix}$ and $\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$, although that's not in the spirit of the question.

Consider $e^{i\alpha}z = (\cos\alpha x - \sin\alpha y) + i(\sin\alpha x + \cos\alpha y)$ then $\left|\mathrm{Re}(e^{i\alpha}z)\right| \leq |z|$ for all values of α and in particular we can choose α to match the signs of the x and y to prove the result in question.

(ii) Consider $(5 - \mathrm{i})^4(1 + \mathrm{i})$, then

$$\begin{aligned}
\arg\left((5 - \mathrm{i})^4(1 + \mathrm{i})\right) &= \arg(5 - i)^4 + \arg(1 + i) \\
&= 4\arg(5 - i) + \arg(1 + i) \\
&= -4\tan^{-1}\frac{1}{5} + \tan^{-1} 1 \\
\\
&= \arg((24 - 10i)^2(1 + i)) \\
&= \arg(4(12 - 5i)^2(1 + i)) \\
&= \arg((119 - 120i)(1 + i)) \\
&= \arg(239 - i) \\
&= -\tan^{-1}\frac{1}{239}
\end{aligned}$$

Therefore $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

Consider $(4 - i)^3(1 + i)(20 - i)$ then

$$\arg\left((4 - i)^3(1 + i)(20 - i)\right) = -3\tan^{-1}\frac{1}{4} + \tan^{-1} 1 - \tan^{-1}\frac{1}{20}$$

$$\begin{aligned}
&= \arg\left((15-8i)(4-i)(1+i)(20-i)\right) \\
&= \arg\left((52-47i)(1+i)(20-i)\right) \\
&= \arg\left((99+5i)(20-i)\right) \\
&= \arg(1985+i) \\
&= \tan^{-1}\frac{1}{1985}
\end{aligned}$$

Therefore $\dfrac{\pi}{4} = 3\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{1}{20} + \tan^{-1}\dfrac{1}{1985}$

Question (2025 STEP III Q8)**(i)** Show that

$$z^{m+1} - \frac{1}{z^{m+1}} = \left(z - \frac{1}{z}\right)\left(z^m + \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right)$$

Hence prove by induction that, for $n \geq 1$,

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right)\sum_{r=1}^{n}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right)$$

Find similarly $z^{2n} - \frac{1}{z^{2n}}$ as a product of $(z + \frac{1}{z})$ and a sum.

(ii) i. By choosing $z = e^{i\theta}$, show that

$$\sin 2n\theta = 2\sin\theta\sum_{r=1}^{n}\cos(2r-1)\theta$$

ii. Use this result, with $n = 2$, to show that $\cos\frac{2\pi}{5} = \cos\frac{\pi}{5} - \frac{1}{2}$.

iii. Use this result, with $n = 7$, to show that $\cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15} + \cos\frac{16\pi}{15} = \frac{1}{2}$.

(iii) Show that $\sin\frac{\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14} = \frac{1}{2}$.

(i)

$$\begin{aligned}
RHS &= \left(z - \frac{1}{z}\right)\left(z^m + \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right) \\
&= z^{m+1} + \frac{1}{z^{m-1}} - z^{m-1} - \frac{1}{z^{m+1}} + z^{m-1} - \frac{1}{z^{m-1}} \\
&= z^{m+1} - \frac{1}{z^{m+1}} \\
&= LHS
\end{aligned}$$

.

Claim: For $n \geq 1$,

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right) \sum_{r=1}^{n} \left(z^{2r-1} + \frac{1}{z^{2r-1}}\right)$$

Proof: (By Induction) Base Case: $(n = 1)$.

$$\begin{aligned} LHS &= z^2 - \frac{1}{z^2} \\ &= (z - \frac{1}{z})(z + \frac{1}{z}) \\ &= (z - \frac{1}{z}) \sum_{r=1}^{1} \left(z + \frac{1}{z}\right) \\ &= (z - \frac{1}{z}) \sum_{r=1}^{1} \left(z^{2r-1} + \frac{1}{z^{2r-1}}\right) \\ &= RHS \end{aligned}$$

as required. Inductive step: Suppose our result is true for some $n = k$, then consider $n = k + 1$.

$$\begin{aligned} RHS &= \left(z - \frac{1}{z}\right) \sum_{r=1}^{k+1} \left(z^{2r-1} + \frac{1}{z^{2r-1}}\right) \\ &= \left(z - \frac{1}{z}\right) \sum_{r=1}^{k} \left(z^{2r-1} + \frac{1}{z^{2r-1}}\right) + \left(z - \frac{1}{z}\right) \left(z^{2k+1} + \frac{1}{z^{2k+1}}\right) \\ &= z^{2k} - \frac{1}{z^{2k}} + \left(z - \frac{1}{z}\right) \left(z^{2k+1} + \frac{1}{z^{2k+1}}\right) \\ &= z^{2k+2} - \frac{1}{z^{2k+2}} \\ &= LHS \end{aligned}$$

.

Therefore if our result is true for $n = k$ is true, it is true for $n = k + 1$. Since it is also true for $n = 1$ it is true for all $n \geq 1$ but the principle of mathematical induction.

Since $z^{m+1} - \frac{1}{z^{m+1}} = \left(z + \frac{1}{z}\right) \left(z^m - \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right)$, we must have $z^{2n} - \frac{1}{z^{2n}} = \left(z + \frac{1}{z}\right) \sum_{r=1}^{n} \left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)$

(ii) i. Since

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right) \sum_{r=1}^{n} \left(z^{2r-1} + \frac{1}{z^{2r-1}}\right)$$

we have

$$e^{2n\theta i} - e^{-2n\theta i} = \left(e^{\theta i} - e^{-\theta i}\right)\sum_{r=1}^{n}\left(e^{(2r-1)\theta i} + e^{-(2r-1)\theta i}\right)$$

$$\Rightarrow \quad 2i\sin 2n\theta = 2i\sin\theta\sum_{r=1}^{n} 2\cos(2r-1)\theta$$

$$\Rightarrow \quad \sin 2n\theta = 2\sin\theta\sum_{r=1}^{n}\cos(2r-1)\theta$$

ii. When $n = 2, \theta = \frac{\pi}{5}$ we have:

$$\begin{aligned}
\sin\frac{4\pi}{5} &= 2\sin\frac{\pi}{5}(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}) \\
\sin\frac{\pi}{5} &= 2\sin\frac{\pi}{5}(\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}) \\
\frac{1}{2} &= \cos\frac{\pi}{5} - \cos\frac{2\pi}{5} \\
\Rightarrow \quad \cos\frac{2\pi}{5} &= \cos\frac{\pi}{5} - \frac{1}{2}
\end{aligned}$$

iii. When $n = 7, \theta = \frac{\pi}{15}$ we have:

$$\begin{aligned}
\sin\frac{14\pi}{15} &= 2\sin\frac{\pi}{15}\sum_{r=1}^{7}\cos(2r-1)\frac{\pi}{15} \\
\Rightarrow \quad \frac{1}{2} &= \cos\frac{\pi}{15} + \cos\frac{3\pi}{15} + \cos\frac{5\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{9\pi}{15} + \cos\frac{11\pi}{15} + \cos\frac{13\pi}{15} \\
&= -\cos\frac{16\pi}{15} + \cos\frac{3\pi}{15} + \cos\frac{5\pi}{15} - \cos\frac{8\pi}{15} + \cos\frac{9\pi}{15} - \cos\frac{4\pi}{15} - \cos\frac{2\pi}{15} \\
&= -\left(\cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15} + \cos\frac{16\pi}{15}\right) + \cos\frac{\pi}{5} + \cos\frac{\pi}{3} + \cos\frac{3\pi}{5} \\
&= -\left(\cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15} + \cos\frac{16\pi}{15}\right) + \frac{1}{2} + \frac{1}{2} \\
\Rightarrow \quad \frac{1}{2} &= cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15} + \cos\frac{16\pi}{15}
\end{aligned}$$

(iii) By using $z = e^{i\theta}$ we have that:

$$z^{2n} - \frac{1}{z^{2n}} = \left(z + \frac{1}{z}\right)\sum_{r=1}^{n}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)$$

$$\Rightarrow \quad e^{2n\theta i} - e^{-2n\theta i} = (e^{\theta i} + e^{-\theta i})\sum_{r=1}^{n}(e^{(2r-1)\theta i} - e^{(2r-1)\theta i})$$

$$\Rightarrow \quad 2i\sin 2n\theta = 2\cos\theta\sum_{r=1}^{n} 2i\sin(2r-1)\theta$$

$$\Rightarrow \quad \sin 2n\theta = 2\cos\theta\sum_{r=1}^{n}\sin(2r-1)\theta$$

When $n = 3, \theta = \frac{\pi}{14}$ we must have:

$$\begin{aligned} \sin\frac{3\pi}{7} &= 2\cos\frac{\pi}{14}(\sin\frac{\pi}{14} + \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14}) \\ &= 2\sin\left(\frac{\pi}{2} - \frac{\pi}{14}\right)(\sin\frac{\pi}{14} + \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14}) \\ &= 2\sin\frac{3\pi}{7}(\sin\frac{\pi}{14} + \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14}) \\ \Rightarrow \qquad \frac{1}{2} &= \sin\frac{\pi}{14} + \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14} \end{aligned}$$

as required.

Question (1988 STEP III Q7)

For $n = 0, 1, 2, \dots,$ the functions y_n satisfy the differential equation

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \omega^2 x^2 y_n = -(2n+1)\omega y_n,$$

where ω is a positive constant, and $y_n \to 0$ and $\mathrm{d}y_n/\mathrm{d}x \to 0$ as $x \to +\infty$ and as $x \to -\infty$. Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = \mathrm{e}^{-\lambda x^2} \qquad \text{and} \qquad y_1(x) = x\mathrm{e}^{-\lambda x^2}$$

for some constant λ, to be determined. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y_m\frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n\frac{\mathrm{d}y_m}{\mathrm{d}x}\right) = 2(m-n)\omega y_m y_n,$$

and deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x)\,\mathrm{d}x = 0.$$

$$\begin{aligned} y_0(x) &= e^{-\lambda x^2} \\ \lim_{x\to\pm\infty} y_0(x) &= 0 \Leftrightarrow \lambda > 0 \\ \lim_{x\to\pm\infty} y_0'(x) &= \lim_{x\to\pm\infty} 2x\lambda e^{-\lambda x^2} \\ &= 0 \Leftrightarrow \lambda > 0 \\ y_0''(x) &= 4x^2\lambda^2 e^{-\lambda x^2} + 2\lambda e^{-\lambda x^2} \end{aligned}$$

$$\begin{aligned} y_0'' - \omega^2 x^2 y_0 + (2\cdot 0 + 1)\omega y_0 &= e^{-\lambda x^2}\left(4x^2\lambda^2 + 2\lambda - \omega^2 x^2 + \omega\right) \\ &= 0 \Leftrightarrow \lambda = \pm\frac{\omega}{2} \end{aligned}$$

Therefore y_0 satisfies if $\lambda = \frac{\omega}{2}$

Similarly for y_1,

$$y_1(x) = xe^{-\lambda x^2}$$

$$\begin{aligned}
\lim_{x\to\pm\infty} y_1(x) &= 0 \Leftrightarrow \lambda > 0 \\
\lim_{x\to\pm\infty} y_1'(x) &= \lim_{x\to\pm\infty}\left(-2x^2\lambda e^{-\lambda x^2} + e^{-\lambda x^2}\right) \\
&= 0 \Leftrightarrow \lambda > 0 \\
y_0''(x) &= e^{-\lambda x^2}\left(4x^3\lambda^2 - 4x\lambda - 2x\lambda\right) \\
&= e^{-\lambda x^2}\left(4x^3\lambda^2 - 6x\lambda\right) \\
y_1'' - \omega^2x^2y_1 + (2\cdot 1+1)\omega y_1 &= e^{-\lambda x^2}\left(4x^3\lambda^2 - 6x\lambda - \omega^2x^3 + 3\omega x\right) \\
&= 0 \Leftrightarrow \lambda = \pm\frac{\omega}{2}
\end{aligned}$$

Therefore y_1 satisfies if $\lambda = \frac{\omega}{2}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(y_m\frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n\frac{\mathrm{d}y_m}{\mathrm{d}x}\right) &= y_m'y_n' + y_my_n'' - y_n'y_m' - y_ny_m'' \\
&= y_my_n'' - y_ny_m'' \\
&= y_m(\omega^2x^2y_n - (2n+1)\omega y_n) - y_n(\omega^2x^2y_m - (2m+1)\omega y_m) \\
&= y_my_n(2m-2n)\omega \\
&= 2(m-n)\omega y_my_n
\end{aligned}$$

Therefore:

$$\begin{aligned}
\int_{-\infty}^{\infty} y_m(x)y_n(x)\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{1}{2(m-n)}\frac{\mathrm{d}}{\mathrm{d}x}\left(y_m\frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n\frac{\mathrm{d}y_m}{\mathrm{d}x}\right)\mathrm{d}x \\
&= \frac{1}{2(m-n)}\left[y_m\frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n\frac{\mathrm{d}y_m}{\mathrm{d}x}\right]_{-\infty}^{\infty} \\
&\to 0
\end{aligned}$$

This condition is known as Orthogonality. In fact this question is talking about a Sturm-Liouville orthogonality condition, in particular for the quantum harmonic oscillator, and the eigenfunctions are related to Hermite polynomials.

> **Question** (1989 STEP II Q7)
> By means of the substitution x^α, where α is a suitably chosen constant, find the general solution for $x > 0$ of the differential equation
>
> $$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - b\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2b+1}y = 0,$$
>
> where b is a constant and $b > -1$. Show that, if $b > 0$, there exist solutions which satisfy $y \to 1$ and $\mathrm{d}y/\mathrm{d}x \to 0$ as $x \to 0$, but that these conditions do not determine a unique solution. For what values of b do these conditions determine a unique solution?

Let $z = x^\alpha$, $\frac{\mathrm{d}z}{\mathrm{d}x} = \alpha x^{\alpha-1}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

$$= \alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}z}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}z}\right)\\ &= \alpha(\alpha-1)x^{\alpha-2}\frac{\mathrm{d}y}{\mathrm{d}z} + \alpha x^{\alpha-1}\frac{\mathrm{d}^2y}{\mathrm{d}z^2}\frac{\mathrm{d}z}{\mathrm{d}x}\\ &= \alpha(\alpha-1)x^{\alpha-2}\frac{\mathrm{d}y}{\mathrm{d}z} + \alpha^2x^{2\alpha-2}\frac{\mathrm{d}^2y}{\mathrm{d}z^2}\end{aligned}$$

$$\begin{aligned}0 &= x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - b\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2b+1}y\\ &= x\left(\alpha(\alpha-1)x^{\alpha-2}\frac{\mathrm{d}y}{\mathrm{d}z} + \alpha^2x^{2\alpha-2}\frac{\mathrm{d}^2y}{\mathrm{d}z^2}\right) - b\left(\alpha x^{\alpha-1}\frac{\mathrm{d}y}{\mathrm{d}z}\right) + x^{2b+1}y\\ &= \alpha^2x^{2\alpha-1}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \left(\alpha(\alpha-1)x^{\alpha-1} - b\alpha x^{\alpha-1}\right)\frac{\mathrm{d}y}{\mathrm{d}z} + x^{2b+1}y\end{aligned}$$

If we set $\alpha = b+1$ the middle term disappears, so we get

$$\begin{aligned} & & 0 &= (b+1)^2x^{2b+1}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + x^{2b+1}y\\ &\Rightarrow & 0 &= (b+1)^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + y\\ &\Rightarrow & y &= A\sin\left(\frac{z}{b+1}\right) + B\cos\left(\frac{z}{b+1}\right)\\ & & &= \boxed{A\sin\left(\frac{x^{b+1}}{b+1}\right) + B\cos\left(\frac{x^{b+1}}{b+1}\right)}\end{aligned}$$

$$\begin{aligned} &\lim_{x\to 0}: & y &\to B\\ & & \frac{\mathrm{d}y}{\mathrm{d}x} &= Ax^b\cos\left(\frac{x^{b+1}}{b+1}\right) - Bx^b\sin\left(\frac{x^{b+1}}{b+1}\right)\\ &b>0: & \frac{\mathrm{d}y}{\mathrm{d}x} &\to 0\end{aligned}$$

So there are infinitely many different solutions with $B = 1$ and A is anything it wants to be.

If $b = 0$ $y' \to A$ so $A = 0$ and unique.

If $b < 0$ $x^b \to \infty$ so we need $A = 0$, unique. However, we also need $y' \to 0$, so we need to check $y' = -x^b\sin\left(\frac{x^{b+1}}{b+1}\right) \to 0$,

$$y' = -x^b\sin\left(\frac{x^{b+1}}{b+1}\right)$$

$$\approx -x^b \left(\frac{x^{b+1}}{b+1} \right)$$
$$= -\frac{x^{2b+1}}{b+1}$$

so we need $2b + 1 > 0 \Rightarrow b > -\frac{1}{2}$.

Therefore the solution is unique on $(-\frac{1}{2}, 0]$

Question (1995 STEP I Q8)

Find functions f, g and h such that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y = \mathrm{h}(x) \qquad (*)$$

is satisfied by all three of the solutions $y = x, y = 1$ and $y = x^{-1}$ for $0 < x < 1$.

If f, g and h are the functions you have found in the first paragraph, what condition must the real numbers a, b and c satisfy in order that

$$y = ax + b + \frac{c}{x}$$

should be a solution of $(*)$?

None

Question (1995 STEP III Q3)

What is the general solution of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

for each of the cases: (i) $k > 1$; (ii) $k = 1$; (iii) $0 < x < 1$? In case (iii) the equation represents damped simple harmonic motion with damping factor k. Let $x(0) = 0$ and let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of successive maxima and minima, so that if x_n is a maximum then x_{n+1} is the next minimum. Show that $|x_{n+1}/x_n|$ takes a value α which is independent of n, and that

$$k^2 = \frac{(\ln \alpha)^2}{\pi^2 + (\ln \alpha)^2}.$$

None

Question (1995 STEP III Q5)

Show that $y = \sin^2(m \sin^{-1} x)$ satisfies the differential equation

$$(1-x^2)y^{(2)} = xy^{(1)} + 2m^2(1-2y),$$

and deduce that, for all $n \geqslant 1$,

$$(1-x^2)y^{(n+2)} = (2n+1)xy^{(n+1)} + (n^2-4m^2)y^{(n)},$$

where $y^{(n)}$ denotes the nth derivative of y.

Derive the Maclaurin series for y, making it clear what the general term is.

Question (1996 STEP II Q8)

Suppose that

$$\mathrm{f}''(x) + \mathrm{f}(-x) = x + 3\cos 2x$$

and $\mathrm{f}(0) = 1$, $\mathrm{f}'(0) = -1$. If $\mathrm{g}(x) = \mathrm{f}(x) + \mathrm{f}(-x)$, find $\mathrm{g}(0)$ and show that $\mathrm{g}'(0) = 0$. Show that

$$\mathrm{g}''(x) + \mathrm{g}(x) = 6\cos 2x,$$

and hence find $\mathrm{g}(x)$. Similarly, if $\mathrm{h}(x) = \mathrm{f}(x) - \mathrm{f}(-x)$, find $\mathrm{h}(x)$ and show that

$$\mathrm{f}(x) = 2\cos x - \cos 2x - x.$$

$$\begin{aligned}
g(0) &= f(0) + f(-0) = 2f(0) = 2 \\
g'(x) &= f'(x) - f'(-x) \\
g'(0) &= f'(0) - f'(-0) = 0 \\
g''(x) &= f''(x) + f''(-x) \\
\Rightarrow \qquad g''(x) + g(x) &= f''(x) + f''(-x) + f(x) + f(-x) \\
&= f''(x) + f(-x) + f''(-x) + f(x) \\
&= x + 3\cos 2x + (-x + 3\cos(-2x)) \\
&= 6\cos 2x
\end{aligned}$$

Considering the homogeneous part, we should expected a solution of the form $g(x) = A\sin x + B\cos x$. Seeking an integrating factor of the form $g(x) = C\cos 2x$ we see that $-4C\cos 2x + C\cos 2x = 6\cos 2x \Rightarrow -3C = 6 \Rightarrow C = -2$. Therefore the general solution is

$$\begin{aligned}
g(x) &= A\sin x + B\cos x - 2\cos 2x \\
g(0) &= B - 2 = 2 \\
g'(0) &= A = 0 \\
\Rightarrow \qquad g(x) &= 4\cos x - 2\cos 2x
\end{aligned}$$

$$h(0) = f(0) - f(-0) = 0$$

$$\begin{aligned}
h'(x) &= f'(x) + f'(-x) \\
h'(0) &= f'(0) + f'(-0) = -2 \\
h''(x) &= f''(x) - f''(-x) \\
\Rightarrow \qquad h''(x) - h(x) &= f''(x) - f''(-x) - (f(x) - f(-x)) \\
&= f''(x) + f(-x) - (f''(-x) + f(x)) \\
&= x + 3\cos 2x - (-x + 3\cos(-2x)) \\
&= 2x
\end{aligned}$$

Considering the homogeneous part, we should expect a solution of the form $Ae^x + Be^{-x}$. For a specific integral, we can take $-2x$, ie

$$\begin{aligned}
h(x) &= Ae^x + Be^{-x} - 2x \\
h(0) &= A + B = 0 \\
h'(0) &= A - B - 2 = -2 \\
\Rightarrow \qquad A &= B = 0 \\
\Rightarrow \qquad h(x) &= -2x
\end{aligned}$$

Therefore

$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) = 2\cos x - \cos 2x - x$

Question (1997 STEP III Q6)

Suppose that y_n satisfies the equations

$$(1-x^2)\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - x\frac{\mathrm{d}y_n}{\mathrm{d}x} + n^2 y_n = 0,$$

$$y_n(1) = 1, \quad y_n(x) = (-1)^n y_n(-x).$$

If $x = \cos\theta$, show that

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}\theta^2} + n^2 y_n = 0,$$

and hence obtain y_n as a function of θ. Deduce that for $|x| \leqslant 1$

$$y_0 = 1, \quad y_1 = x,$$

$$y_{n+1} - 2xy_n + y_{n-1} = 0.$$

Question (1999 STEP I Q7)

Show that $\sin(k \sin^{-1} x)$, where k is a constant, satisfies the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + k^2y = 0. \qquad (*)$$

In the particular case when $k = 3$, find the solution of equation $(*)$ of the form

$$y = Ax^3 + Bx^2 + Cx + D,$$

that satisfies $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ at $x = 0$. Use this result to express $\sin 3\theta$ in terms of powers of $\sin\theta$.

Question (1999 STEP III Q8)

The function $y(x)$ is defined for $x \geq 0$ and satisfies the conditions

$$y = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \text{at } x = 0.$$

When x is in the range $2(n-1)\pi < x < 2n\pi$, where n is a positive integer, $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + n^2y = 0.$$

Both y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ are continuous at $x = 2n\pi$ for $n = 0,\ 1,\ 2,\ \ldots$.

(i) Find $y(x)$ for $0 \leq x \leq 2\pi$.

(ii) Show that $y(x) = \frac{1}{2}\sin 2x$ for $2\pi \leq x \leq 4\pi$, and find $y(x)$ for all $x \geq 0$.

(iii) Show that

$$\int_0^\infty y^2 \,\mathrm{d}x = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

Question (2001 STEP I Q8)

Given that $y = x$ and $y = 1 - x^2$ satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (x)\frac{\mathrm{d}y}{\mathrm{d}x} + (x)y = 0\ ,$$

show that $(x) = -2x(1+x^2)^{-1}$ and $(x) = 2(1+x^2)^{-1}$. Show also that $ax+b(1-x^2)$ satisfies the differential equation for any constants a and b. Given instead that $y = \cos^2(\frac{1}{2}x^2)$ and $y = \sin^2(\frac{1}{2}x^2)$ satisfy the equation $(*)$, find (x) and (x).

$$y = x$$

$$\begin{aligned}
y' &= 1 \\
y'' &= 0 \\
\Rightarrow \qquad 0 &= 0 + p(x) + xq(x) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
y &= 1 - x^2 \\
y' &= -2x \\
y'' &= -2 \\
\Rightarrow \qquad 0 &= -2 - 2xp(x) + (1 - x^2)q(x) \qquad (2)
\end{aligned}$$

$$\begin{aligned}
2x * (1) + (2): \qquad 2 &= (2x^2 + 1 - x^2)q(x) \\
\Rightarrow \qquad q(x) &= 2(1 + x^2)^{-1} \\
\Rightarrow \qquad p(x) &= -2x(1 + x^2)^{-1} \qquad \text{(by (1))}
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(ax + b(1 - x^2)\right) + p(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(ax + b(1 - x^2)\right) + q(x)\left(ax + b(1 - x^2)\right) \\
&= a\frac{\mathrm{d}^2 x}{\mathrm{d}x^2} + b\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(1 - x^2\right) + ap(x)\frac{\mathrm{d}x}{\mathrm{d}x} + bp(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(1 - x^2\right) + aq(x)x + bq(x)(1 - x^2) \\
&= a\left(\frac{\mathrm{d}^2 x}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}x}{\mathrm{d}x} + q(x)x\right) + b\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(1 - x^2\right) + p(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(1 - x^2\right) + q(x)(1 - x^2)\right) = 0
\end{aligned}$$

$$\begin{aligned}
y &= \cos^2(\tfrac{1}{2}x^2) = \frac{1}{2}\left(1 + \cos(x^2)\right) \\
y' &= -x\sin(x^2) \\
y'' &= -2x^2\cos(x^2) - \sin(x^2) \\
\Rightarrow \qquad 0 &= -2x^2\cos(x^2) - \sin(x^2) + p(x)(-x\sin(x^2)) + \frac{1}{2}\left(1 + \cos(x^2)\right)q(x) \\
\Rightarrow \qquad 2x^2\cos(x^2) + \sin(x^2) &= -x\sin(x^2)p(x) + \frac{1}{2}(1 + \cos(x^2))q(x) \qquad (3)
\end{aligned}$$

$$\begin{aligned}
y &= \sin^2(\tfrac{1}{2}x^2) = \frac{1}{2}\left(1 - \cos(x^2)\right) \\
y' &= x\sin(x^2) \\
y'' &= 2x^2\cos(x^2) + \sin(x^2) \\
\Rightarrow \qquad 0 &= 2x^2\cos(x^2) + \sin(x^2) + p(x)x\sin(x^2) + \frac{1}{2}\left(1 - \cos(x^2)\right)q(x) \\
\Rightarrow \qquad -2x^2\cos(x^2) - \sin(x^2) &= p(x)x\sin(x^2) + \frac{1}{2}\left(1 - \cos(x^2)\right)q(x) \qquad (4) \\
(3) + (4): \qquad 0 &= q(x) \\
\Rightarrow \qquad p(x) &= -\frac{2x^2\cos(x^2) + \sin(x^2)}{x\sin(x^2)}
\end{aligned}$$

Question (2007 STEP III Q8)**(i)** Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation. Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

Question (2008 STEP III Q6)

In this question, p denotes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Given that

$$y = p^2 + 2xp\,,$$

show by differentiating with respect to x that

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -2 - \frac{2x}{p}.$$

Hence show that $x = -\frac{2}{3}p + Ap^{-2}$, where A is an arbitrary constant. Find y in terms of x if $p = -3$ when $x = 2$.

(ii) Given instead that

$$y = 2xp + p\ln p\,,$$

and that $p = 1$ when $x = -\frac{1}{4}$, show that $x = -\frac{1}{2}\ln p - \frac{1}{4}$ and find y in terms of x.

Question (2009 STEP III Q2)**(i)** Let $y = \sum_{n=0}^{\infty} a_n x^n$, where the coefficients a_n are independent of x and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} ,$$

and write down a similar expression for y''. Write out explicitly each of the three series as far as the term containing a_3.

(ii) It is given that y satisfies the differential equation

$$xy'' - y' + 4x^3 y = 0 .$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that $a_1 = 0$. Show that, for $n \geq 4$,

$$a_n = -\frac{4}{n(n-2)} a_{n-4} ,$$

and that, if $a_0 = 1$ and $a_2 = 0$, then $y = \cos(x^2)$. Find the corresponding result when $a_0 = 0$ and $a_2 = 1$.

(i) Let $y = \sum_{n=0}^{\infty} a_n x^n$ then

$$\begin{aligned} y' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(a_n x^n\right) \\ &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned}$$

$$\begin{aligned} y'' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\ &= \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(n a_n x^{n-1}\right) \\ &= \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ y' &= a_1 + 2a_2x + 3a_3x^2 + \cdots \\ y'' &= 2a_2 + 6a_3x + \cdots \end{aligned}$$

(ii)

$$\begin{aligned} 0 &= xy'' - y' + 4x^3y \\ &= x\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=1}^{\infty} na_nx^{n-1} + 4x^3\sum_{n=0}^{\infty} a_nx^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} - \sum_{n=1}^{\infty} na_nx^{n-1} + \sum_{n=0}^{\infty} 4a_nx^{n+3} \\ &= \sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} - \sum_{n=1}^{\infty} na_nx^{n-1} + \sum_{n=4}^{\infty} 4a_{n-4}x^{n-1} \\ &= \sum_{n=4}^{\infty} \left(n(n-1)a_n - na_n + 4a_{n-4}\right)x^{n-1} + 2a_2x + 6a_3x^2 - a_1 - 2a_2x - 3a_3x^2 \\ &= \sum_{n=4}^{\infty} \left(n(n-2)a_n + 4a_{n-4}\right)x^{n-1} + 3a_3x^2 - a_1 \end{aligned}$$

Therefore since all coefficients are 0, $a_1 = 0$, $a_3 = 0$ and $a_n = -\dfrac{4}{n(n-2)}a_{n-4}$.

If $a_0 = 1, a_2 = 0$, and since $a_1 = 0, a_3 = 0$ the only values which will take non-zero value are a_{4k}. We can compute these values as: $a_{4k} = -\frac{4}{(4k)(4k-2)}a_{4k-4} = \frac{1}{2k(2k-1)}a_{4k-r}$ so $a_{4k} = \frac{(-1)^k}{(2k)!}$, which are precisely the coefficients in the expansion $\cos x^2$.

If $a_0 = 0, a_2 = 1$ then since $a_1 = 0, a_3 = 0$ the only values which take non-zero values are a_{4k+2} we can compute these values as:

$a_{4k+2} = -\frac{4}{(4k+2)(4k)}a_{4k-2} = -\frac{1}{(2k+1)2k}a_{4k-2}$ so we can see that $a_{4k+2} = \frac{(-1)^k}{(2k+1)!}$ precisely the coefficients of $\sin x^2$

Question (2009 STEP III Q7)**(i)** The functions $f_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$f_0(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad f_{n+1}(x) = \frac{\mathrm{d}f_n(x)}{\mathrm{d}x}\,.$$

Prove, for $n \geq 1$, that

$$(1+x^2)f_{n+1}(x) + 2(n+1)xf_n(x) + n(n+1)f_{n-1}(x) = 0\,.$$

(ii) The functions $\mathbb{P}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathbb{P}_n(x) = (1+x^2)^{n+1}f_n(x)\,.$$

Find expressions for $\mathbb{P}_0(x)$, $\mathbb{P}_1(x)$ and $\mathbb{P}_2(x)$.

Prove, for $n \geq 0$, that

$$\mathbb{P}_{n+1}(x) - (1+x^2)\frac{\mathrm{d}\mathbb{P}_n(x)}{\mathrm{d}x} + 2(n+1)x\mathbb{P}_n(x) = 0\,,$$

and that $\mathbb{P}_n(x)$ is a polynomial of degree n.

Question (2009 STEP III Q10)

A light spring is fixed at its lower end and its axis is vertical. When a certain particle P rests on the top of the spring, the compression is d. When, instead, P is dropped onto the top of the spring from a height h above it, the compression at time t after P hits the top of the spring is x. Obtain a second-order differential equation relating x and t for $0 \leq t \leq T$, where T is the time at which P first loses contact with the spring. Find the solution of this equation in the form

$$x = A + B\cos(\omega t) + C\sin(\omega t)\,,$$

where the constants A, B, C and ω are to be given in terms of d, g and h as appropriate. Show that

$$T = \sqrt{d/g}\,\left(2\pi - 2\arctan\sqrt{2h/d}\,\right)\,.$$

Question (2010 STEP I Q6)

Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

Question (2010 STEP III Q10)

A small bead B, of mass m, slides without friction on a fixed horizontal ring of radius a. The centre of the ring is at O. The bead is attached by a light elastic string to a fixed point P in the plane of the ring such that $OP = b$, where $b > a$. The natural length of the elastic string is c, where $c < b - a$, and its modulus of elasticity is λ. Show that the equation of motion of the bead is

$$ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta+\phi)\,,$$

where $\theta = \angle BPO$ and $\phi = \angle BOP$.

Given that θ and ϕ are small, show that $a(\theta+\phi) \approx b\theta$. Hence find the period of small oscillations about the equilibrium position $\theta = \phi = 0$.

Question (2011 STEP III Q1)**(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right)u = 0\,.$$

(ii) Show that substituting$y = z\mathrm{e}^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\frac{\mathrm{d}z}{\mathrm{d}x}$. Find z and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x}\,,$$

where A and B are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

(i)

$$\begin{aligned}
&& 0 &= \frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right)u \\
\Rightarrow && \int \frac{1}{u}\mathrm{d}u &= \int 1 + \frac{1}{x+1}\mathrm{d}x \\
\Rightarrow && \ln|u| &= x + \ln|x+1| + C \\
\Rightarrow && u &= A(x+1)e^x
\end{aligned}$$

(ii) If $y = ze^{-x}$, $y' = (z'-z)e^{-x}$, $y'' = (z''-2z'+z)e^{-x}$

$$\begin{aligned}
&& 0 &= (x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y \\
y = ze^{-x}: && 0 &= (x+1)\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right)e^{-x} + x\left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right)e^{-x} - ze^{-x} \\
&& &= (x+1)\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - (x+2)\frac{\mathrm{d}z}{\mathrm{d}x} \\
\Rightarrow && \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right) &= \left(\frac{x+2}{x+1}\right)\frac{\mathrm{d}z}{\mathrm{d}x}
\end{aligned}$$

Therefore $\frac{\mathrm{d}z}{\mathrm{d}x} = A(x+1)e^x$ and so

$$z = A\int (x+1)e^x \mathrm{d}x$$

$$
\begin{aligned}
&= A\left([(x+1)e^x] - \int e^x \mathrm{d}x\right) \\
&= A(x+1)e^x - Ae^x + B \\
y &= Ax + Be^{-x}
\end{aligned}
$$

(iii) We have found the complementary solution. To find a particular integral consider $y = ax^2 + bx + c$, then $y' = 2ax + b, y'' = 2a$ and we have

$$
\begin{aligned}
& x^2 + 2x + 1 = 2a(x+1) + x(2ax+b) - (ax^2 + bx + c) \\
\Rightarrow \quad & x^2 + 2x + 1 = ax^2 + 2ax + 2a - c \\
\Rightarrow \quad & a = 1, c = 1
\end{aligned}
$$

so the general solution should be

$$
y = Ax + Be^{-x} + x^2 + 1
$$

Question (2012 STEP III Q1)

Given that $z = y^n \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$
\frac{\mathrm{d}z}{\mathrm{d}x} = y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).
$$

(i) Use the above result to show that the solution to the equation

$$
\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y} \qquad (y > 0)
$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ is $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$.

(ii) Find the solution to the equation

$$
\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0
$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

$$
\begin{aligned}
z &= y^n \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}x} &= ny^{n-1}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + y^n \cdot 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) \\
&= y^{n-1}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)
\end{aligned}
$$

(i) Let $z = y(y')^2$, then

$$
\begin{aligned}
& & \frac{\mathrm{d}z}{\mathrm{d}x} &= y'\sqrt{y} \\
& & &= \sqrt{z} \\
&\Rightarrow & \int z^{-1/2}\mathrm{d}z &= x + C \\
&\Rightarrow & 2\sqrt{z} &= x + C \\
x = 0, z = 0 : & & C &= 0 \\
&\Rightarrow & y(y')^2 &= \frac{1}{4}x^2 \\
&\Rightarrow & \sqrt{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}x \\
&\Rightarrow & \int \sqrt{y}\mathrm{d}y &= \int \frac{1}{2}x\mathrm{d}x \\
&\Rightarrow & \frac{2}{3}y^{3/2} &= \frac{1}{4}x^2 + K \\
x = 0, y = 1 : & & K &= \frac{2}{3} \\
&\Rightarrow & y &= \left(\frac{3}{8}x^2 + 1\right)^{2/3}
\end{aligned}
$$

(ii) Let $z = y^{-2}(y')^2$

$$
\begin{aligned}
& & \frac{\mathrm{d}z}{\mathrm{d}x} &= y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}\left(-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) \\
& & &= y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}2y^2 \\
& & &= 2y^{-1}(y') = 2\sqrt{z} \\
&\Rightarrow & 2\sqrt{z} &= 2x + C \\
x = 0, z = 0 : & & C &= 0 \\
&\Rightarrow & z &= x^2 \\
&\Rightarrow & \frac{\mathrm{d}y}{\mathrm{d}x} &= xy \\
&\Rightarrow & \ln|y| &= \frac{1}{2}x^2 + K \\
x = 0, y = 1; & & K &= 0 \\
&\Rightarrow & y &= e^{\frac{1}{2}x^2}
\end{aligned}
$$

Question (2012 STEP III Q7)

A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z)\,, \qquad \dot{z} = -\dot{y} - 3z\,,$$

where the dot denotes differentiation with respect to t. Obtain a second order differential equation for y and hence derive the solution

$$y = A\mathrm{e}^{-t} + B\mathrm{e}^{-6t}\,, \qquad z = \tfrac{1}{2}A\mathrm{e}^{-t} - 2B\mathrm{e}^{-6t}\,,$$

where A and B are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \le t \le 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t - n)\,,$$

provided c takes a particular value that you should find.

Question (2013 STEP III Q7)**(i)** Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}y^4\,.$$

Show by differentiation that E is constant and deduce that $|y(x)| \le 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v\,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \le 0$ for $x \ge 0$ and deduce that $\cosh v(x) \le \frac{5}{3}$ for $x \ge 0$.

(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \le \frac{5}{4}$ for $x \ge 0$.

Question (2013 STEP III Q9)

A sphere of radius R and uniform density ρ_{s} is floating in a large tank of liquid of uniform density ρ. Given that the centre of the sphere is a distance x above the level of the liquid, where $x < R$, show that the volume of liquid displaced is

$$\frac{\pi}{3}(2R^3 - 3R^2x + x^3)\,.$$

The sphere is acted upon by two forces only: its weight and an upward force equal in magnitude to the weight of the liquid it has displaced. Show that

$$4R^3\rho_{\mathrm{s}}(g + \ddot{x}) = (2R^3 - 3R^2x + x^3)\rho g\,.$$

Given that the sphere is in equilibrium when $x = \frac{1}{2}R$, find ρ_{s} in terms of ρ. Find, in terms of R and g, the period of small oscillations about this equilibrium position.

None

Question (2014 STEP III Q10)

Two particles X and Y, of equal mass m, lie on a smooth horizontal table and are connected by a light elastic spring of natural length a and modulus of elasticity λ. Two more springs, identical to the first, connect X to a point P on the table and Y to a point Q on the table. The distance between P and Q is $3a$.

Initially, the particles are held so that $XP = a$, $YQ = \frac{1}{2}a$, and $PXYQ$ is a straight line. The particles are then released. At time t, the particle X is a distance $a + x$ from P and the particle Y is a distance $a + y$ from Q. Show that

$$m\frac{\dot{d}^2x}{\dot{d}t^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving $\frac{\dot{d}^2y}{\dot{d}t^2}$. Deduce that

$$x - y = A\cos\omega t + B\sin\omega t$$

where A and B are constants to be determined and $ma\omega^2 = \lambda$. Find a similar expression for $x + y$. Show that Y will never return to its initial position.

Question (2018 STEP III Q3)

Show that the second-order differential equation

$$x^2y'' + (1 - 2p)x\,y' + (p^2 - q^2)\,y = f(x)\,,$$

where p and q are constants, can be written in the form

$$x^a\big(x^b(x^cy)'\big)' = f(x)\,, \qquad (*)$$

where a, b and c are constants.

(i) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1 - 2p)xy' + (p^2 - q^2)y = 0$$

in the different cases that arise according to the values of p and q.

(ii) Use $(*)$ to derive the general solution of the equation

$$x^2y'' + (1 - 2p)xy' + p^2y = x^n$$

in the different cases that arise according to the values of p and n.

Consider $x^a\big(x^b(x^cy)'\big)'$ then

$$\begin{aligned} x^a\big(x^b(x^cy)'\big)' &= x^a\big(bx^{b-1}(x^cy)' + x^b(x^cy)''\big) \\ &= x^a\big(bx^{b-1}(cx^{c-1}y + x^cy') + x^b(c(c-1)x^{c-2}y + 2cx^{c-1}y' + x^cy''\big) \\ &= x^{a+b+c}y'' + (2cx^{c-1+b+a} + bx^{c+b-1+a})y' + (c(b + c - 1))x^{a+b+c-2}y \end{aligned}$$

So we need:

$$\begin{cases} a+b+c &= 2 \\ 2c+b &= 1-2p \\ c(b+c-1) &= p^2-q^2 \end{cases}$$

$$\begin{aligned} \Rightarrow \quad && c((1-2p)-2c+c-1) = p^2-q^2 \\ \Rightarrow \quad && c^2+2pc = q^2-p^2 \end{aligned}$$

Question (1987 STEP III Q6)

The functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 5y = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} + ax - 2y = 2\cos t,$$

subject to $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ at $t = 0$. Solve these equations for x and y in the case when $a = 1$. Without solving the equations explicitly, state briefly how the form of the solutions for x and y if $a > 1$ would differ from the form when $a = 1$.

Letting $\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -2 & 5 \\ -a & 2 \end{pmatrix}$ then our differential equation is $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} 0 \\ 2\cos t \end{pmatrix}$.

Looking at the eigenvalues of $\mathbf{A}$, we find:

$$\begin{aligned} \det \begin{pmatrix} -2-\lambda & 5 \\ -a & 2-\lambda \end{pmatrix} &= (\lambda^2-4)+5a \\ &= \lambda^2+5a-4 \end{aligned}$$

Therefore if $a = 1$, $\lambda = \pm i$.

In which case we should expect the complementary solutions to be of the form $\mathbf{x} = \begin{pmatrix} A\sin t + B\cos t \\ C\sin t + D\cos t \end{pmatrix}$. The first equation tells us that $(A-5D+B)\cos t + (-B+5C)\sin t = 0$ so the complementary solution is:$\mathbf{x} = \begin{pmatrix} 5(D-C)\sin t + 5C\cos t \\ C\sin t + D\cos t \end{pmatrix}$.

Looking for a particular integral, we should expect to try something like $\mathbf{x} = \begin{pmatrix} Et\cos t + Ft\sin t \\ Gt\cos t + Ht\sin t \end{pmatrix}$ and we find

Question (2019 STEP III Q1)

The coordinates of a particle at time t are x and y. For $t \geq 0$, they satisfy the pair of coupled differential equations

$$\begin{cases} \dot{x} &= -x - ky \\ \dot{y} &= x - y \end{cases}$$

where k is a constant. When $t = 0$, $x = 1$ and $y = 0$.

(i) Let $k = 1$. Find x and y in terms of t and sketch y as a function of t. Sketch the path of the particle in the x-y plane, giving the coordinates of the point at which y is greatest and the coordinates of the point at which x is least.

(ii) Instead, let $k = 0$. Find x and y in terms of t and sketch the path of the particle in the x-y plane.

(i) Let $k = 1$, then

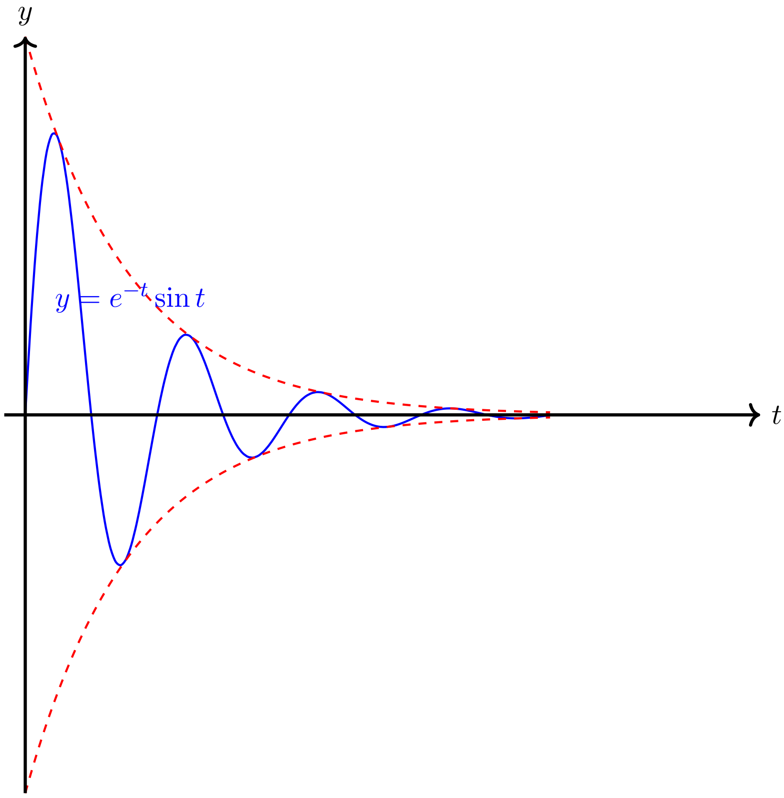
$$\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= x - y \\ \dot{x} - \dot{y} &= -2x \\ \ddot{x} &= -\dot{x} - \dot{y} \\ &= -\dot{x} - (\dot{x} + 2x) \\ &= -2\dot{x} - 2x \\ \dot{x} + \dot{y} &= -2y \\ \ddot{y} &= \dot{x} - \dot{y} \\ &= -2y - 2\dot{y} \end{aligned}$$

So we have an auxiliary equation for x and y which is $\lambda^2+2\lambda+2 = 0 \Rightarrow \lambda = -1 \pm i$.

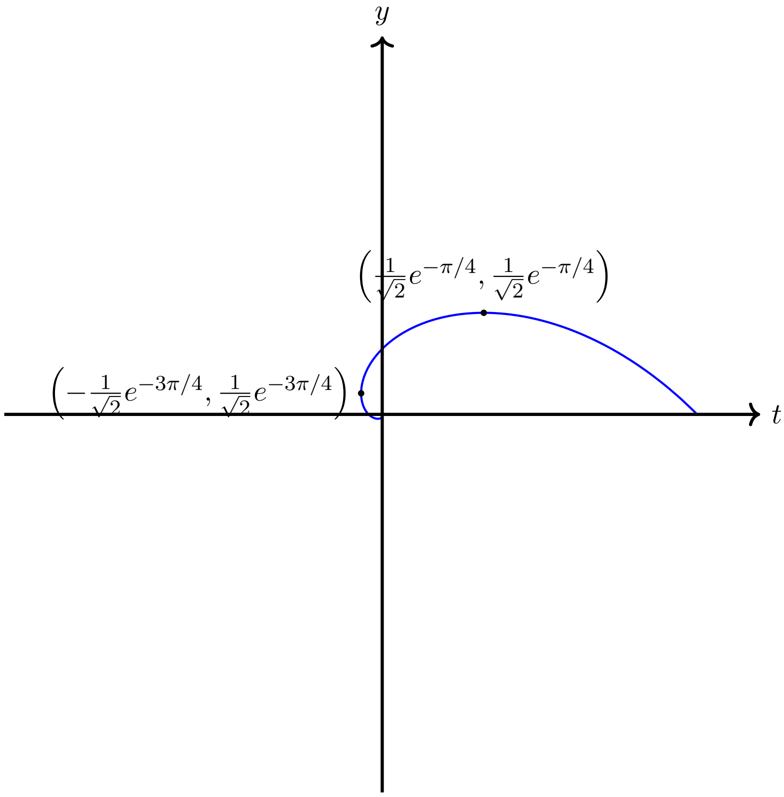
Therefore $x = Ae^{-t}\cos t + Be^{-t}\sin t, y = Ce^{-t}\cos t + De^{-t}\sin t$. We also must have that, $A = 1, C = 0$, so $x = e^{-t}\cos t + Be^{-t}\sin t$ and $y = De^{-t}\sin x$.

$$\begin{aligned} \dot{y} &= -De^{-t}\sin t + De^{-t}\cos t \\ &= e^{-t}\cos x + Be^{-t}\sin t - De^{-t}\sin t \end{aligned}$$

therefore $B = 0, D = 1$ and $x = e^{-t}\cos t, y = e^{-t}\sin t$



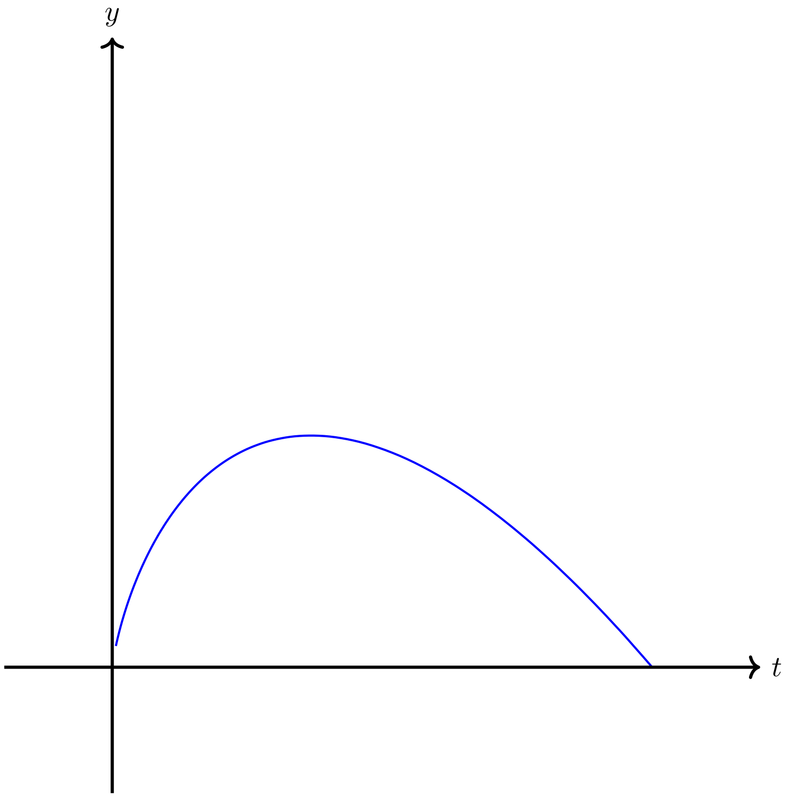
$$y = e^{-t}\sin t$$
$$\dot{y} = -e^{-t}\sin t + e^{-t}\cos t$$
$$\dot{x} = e^{-t}\cos t - e^{-t}\sin t$$



(ii)

$$\dot{x} = -x$$
$$\dot{y} = x - y$$

So $x = e^{-t}$. $\dot{y} + y = e^{-t}$ so $y = (t + B)e^{-t}$ and so $y = te^{-t}$.



Question (2025 STEP II Q7)

The differential equation
$$\frac{d^2x}{dt^2} = 2x\frac{dx}{dt}$$
describes the motion of a particle with position $x(t)$ at time t. At $t = 0$, $x = a$, where $a > 0$.

(i) Solve the differential equation in the case where $\frac{dx}{dt} = a^2$ when $t = 0$. What happens to the particle as t increases from 0?

(ii) Solve the differential equation in the case where $\frac{dx}{dt} = a^2 + p$ when $t = 0$, where $p > 0$. What happens to the particle as t increases from 0?

(iii) Solve the differential equation in the case where $\frac{dx}{dt} = a^2 - q^2$ when $t = 0$, where $q > 0$. What happens to the particle as t increases from 0? Give conditions on a and q for the different cases which arise.

Let $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ and notice that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(v)\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$. Also notice that:

$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}x} &= 2xv \\ \Rightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}x} &= 2x \\ \Rightarrow \qquad v &= x^2 + C \\ \Rightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t} &= x^2 + C \end{aligned}$$

(i) When $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2$ so $C = 0$, therefore $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 \Rightarrow t = -x^{-1} + k$ and so $k = a^{-1}$ and $x = \frac{a}{1-at}$. As t increases from 0 the particle heads to infinity at an increasing rate, 'reaching' infinity around $t = \frac{1}{a}$

(ii) When $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2 + p$ so $C = p$. Therefore $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + p \Rightarrow t = \frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{x}{\sqrt{p}}\right) + c$. When $c = -\frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{a}{\sqrt{p}}\right)$, so

$$\begin{aligned} t &= \frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{x}{\sqrt{p}}\right) - \frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{a}{\sqrt{p}}\right) \\ &= \frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{\sqrt{p}(x-a)}{\sqrt{p}-ax}\right) \\ \Rightarrow \qquad \frac{\sqrt{p}(x-a)}{\sqrt{p}-ax} &= \tan(\sqrt{p}t) \\ \Leftrightarrow \qquad \sqrt{p}(x-a) &= \tan(\sqrt{p}t)(\sqrt{p}-ax) \\ \Leftrightarrow \qquad x(\sqrt{p} + a\tan(\sqrt{p}t)) &= \sqrt{p}(\tan(\sqrt{p}t) + a) \\ \Leftrightarrow \qquad x &= \frac{\sqrt{p}(\tan(\sqrt{p}t) + a)}{\sqrt{p} + a\tan(\sqrt{p}t)} \end{aligned}$$

The particle heads to $\frac{\sqrt{p}}{a}$.

(iii) When $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = a^2 - q^2$ so $C = -q^2$. Therefore

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x^2 - q^2 \\ \Rightarrow \qquad \int \mathrm{d}t &= \int \frac{1}{(x-q)(x+q)}\mathrm{d}x \\ &= \frac{1}{2q}\int\left(\frac{1}{x-q} - \frac{1}{x+q}\right)\mathrm{d}x \\ &= \frac{1}{2q}\left(\ln(x-q) - \ln(x+q)\right) \\ &= \frac{1}{2q}\ln\left(\frac{x-q}{x+q}\right) \\ \Rightarrow \qquad \frac{x-q}{x+q} &= Ae^{2qt} \end{aligned}$$

$$\underbrace{\Rightarrow}_{t=0} \qquad A = \frac{a-q}{a+q}$$

$$\Rightarrow \qquad x - q = \frac{a-q}{a+q}e^{2qt}(x+q)$$

$$\Leftrightarrow \qquad x\left(1 - \frac{a-q}{a+q}e^{2qt}\right) = q\left(1 + \frac{a-q}{a+q}e^{2qt}\right)$$

$$\Leftrightarrow \qquad x = q\frac{1 + \frac{a-q}{a+q}e^{2qt}}{1 - \frac{a-q}{a+q}e^{2qt}}$$